REGULARITY RESULTS FOR FUNCTIONALS WITH GENERAL GROWTH

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In this talk I will present some results on functionals with general growth, obtained in collaboration with L. Diening and A. Verde.

Let φ be a convex, C^1-function and consider the functional:

$$\mathcal{F}(\mathbf{u}) = \int_\Omega \varphi(|\nabla \mathbf{u}|)\, dx \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\mathbf{u} : \Omega \to \mathbb{R}^N$.

The associated Euler Lagrange system is

$$-\operatorname{div}\left(\varphi'(|\nabla \mathbf{u}|)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right) = 0 \tag{0.2}$$

In a fundamental paper K. Uhlenbeck [7] proved everywhere $C^{1,\alpha}$-regularity for local minimizers of the p-growth functional with $p \geq 2$. Later on a large number of generalizations have been made. The case $1 < p < 2$ was considered by Acerbi and Fusco [1] where also the dependence of the functional from x and $\mathbf{u}$ was investigated. In a recent paper [6] Marcellini and Papi proved Lipschitz regularity for local minimizers of functionals with growth conditions general enough to embrace linear and exponential ones.

We prove the $C^{1,\alpha}$-regularity for functionals with φ-growth giving the decay estimate of the excess functional:

$$\Phi(\mathbf{u}, B) = \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2\, dx \tag{0.3}$$

where $\mathbf{V}(\mathbf{Q}) = \sqrt{\varphi'(|\mathbf{Q}|)/|\mathbf{Q}|}\,\mathbf{Q}$ and $B \subset \Omega$ is a ball. To this aim, we make suitable assumptions on the function φ in order to ensure the continuity of the second derivatives of φ. In particular, the case of slow growth and fast growth are ruled out.

Our main theorem is the following:

Theorem 0.1. *Let $\mathbf{u} \in W^{1,\varphi}_{\text{loc}}(\Omega)$ be a local minimizer of (0.1), where φ satisfies suitable assumptions. Then $\mathbf{V}(\nabla \mathbf{u})$ and $\nabla \mathbf{u}$ are locally α-Hölder continuous for some $\alpha > 0$.*

We present a unified approach to the superquadratic and subquadratic p-growth, also considering more general functions than the powers.

As an application, we prove Lipschitz regularity for local minimizers of asymptotically convex functionals in a C^2 sense.

Coming to a general vectorial case, partial regularity comes into the play, as shown in the famous counterexamples of Necas, and also Sverak & Yan. Partial regularity asserts the pointwise regularity of solutions/minimizers, in an open subset whose complement is negligible. The proof of partial regularity compares the original solution $\mathbf{u}$ in a ball with the solution $\mathbf{h}$ in the same ball of the linearized elliptic system with constant coefficients. The comparison map $\mathbf{h}$ is smooth, and enjoys good a-priori estimates. The idea is to establish conditions in order to let $\mathbf{u}$ inherit the regularity estimates of $\mathbf{h}$; for example, $\mathbf{u}$ and $\mathbf{h}$ should be close enough to each other in some integral sense. This is achieved if the original system is "close enough" to the linearized one. Such a linearization idea finds its origins in Geometric Measure Theory, and more precisely in the pioneering work of De Giorgi on minimal surfaces, and of Almgren for minimizing varifolds, and was first implemented by Morrey and Giusti & Miranda for the case of quasilinear systems. Hildebrandt & Kaul & Widman studied partial regularity in the setting of harmonic mappings and related elliptic systems. For the completely non-linear case we have the indirect method via blow-up techniques, implemented originally in the papers of Morrey and Giusti & Miranda, and then recovered directly for the quasiconvex case by Evans, Acerbi, Fusco, Hutchinson, and Hamburger. Another technique is the "A-approximation method", once again first introduced in the setting of Geometric Measure Theory by Duzaar & Steffen and applied to partial regularity for elliptic systems and functionals by Duzaar & Gastel & Grotowski. This method re-exploits the original ideas that De Giorgi introduced in his treatment of minimal surfaces, providing a neat and elementary proof of partial regularity. The linearization is implemented via a suitable variant, for systems with constant coefficients, of the classical "Harmonic approximation lemma" of De Giorgi.

For the p-laplacean system with right-hand side of critical growth, Duzaar and Mingione in [5] proved the $C^{1,\alpha}$ partial regularity via the p-harmonic approximation lemma, that is a nonlinear generalization of the A-harmonic one to $p \neq 2$.

When dealing with general convex function the blow up technique doesn't work so we are forced to find an analogous of the p-harmonic approximation lemma for general convex function, the φ-harmonic approximation lemma.

Lemma 0.2 (φ-harmonic approximation lemma)**.** *Let φ satisfy Assumption* **??**. *For every $\varepsilon > 0$ and $\theta \in (0,1)$ there exists $\delta > 0$ which only depends on ε, θ, and the characteristics of φ such that the following holds. Let $B \subset$ be a ball and let $\widetilde{B}$ denote either B or $2B$. If $\mathbf{u} \in W^{1,\varphi}(\widetilde{B})$ is* almost φ-harmonic *on a ball $B \subset$ in the sense that*

$$\fint_B \varphi'(|\nabla \mathbf{u}|)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\nabla \boldsymbol{\xi}\,dx \leq \delta\bigg(\fint_{\widetilde{B}} \varphi(|\nabla \mathbf{u}|)\,dx + \varphi\big(\|\nabla \boldsymbol{\xi}\|_\infty\big)\bigg) \tag{0.4}$$

for all $\boldsymbol{\xi} \in C_0^\infty(B)$, then the unique φ-harmonic map $\mathbf{h} \in W^{1,\varphi}(B)$ with $\mathbf{h} = \mathbf{u}$ on ∂B satisfies

$$\bigg(\fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h})|^{2\theta}\,dx\bigg)^{\frac{1}{\theta}} < \varepsilon \fint_{\widetilde{B}} \varphi(|\nabla \mathbf{u})|\,dx. \tag{0.5}$$

where $\mathbf{V}(\mathbf{Q}) = \sqrt{\frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}}\mathbf{Q}$.

First of all our definition of *almost φ-harmonic* slightly differs from the original definition of *almost p-harmonic* from [5]. However, as it is easily seen, our definition is weaker; so any *almost p-harmonic* function in the sense of [5] is *almost φ-harmonic* for $\varphi(t) = \frac{1}{p}t^p$

in the sense of (0.4). The reason for chosing this version of *almost harmonic* is, that (0.4) has very good scaling properties.

The main tool in the proof of the previous lemma is a "new" Lipschitz approximation of Sobolev functions that was first introduced by Acerbi & Fusco and then revisited and improved by Diening, Malék and Steinhauer.

As an application of this method, we consider φ-harmonic systems with critical growth and prove a partial regularity result for the solution. Let us observe that in the case of powers we improve the $C^{1,\alpha}$ partial regularity in two different points. From one side, we start with a weaker "smallness" assumption, on the other hand, we give a larger range for the exponent α. In addition, using the closeness of the gradients and not just of the functions the proof shortened very much.

References

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