LECTURE NOTES ON ASPECTS OF REGULARITY THEORY OF MINIMIZERS OF MULTIPLE INTEGRALS

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Abstract. The main aim of the lectures was to discuss the higher differentiability of the minimizers of the integrals $\int_\Omega F(Dv)$ and $\int_\Omega F(X,v,Dv)$ where $v:\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$. If n,$N>1$ then it falls under multi-dimensional vectorial case.

Introduction

The notes have been divided into six sections. In the first section an introduction to the Hilbert´s 19th Problem has been done, with discussion on the regularity of the minimizer of the integrals $\int_\Omega F(Dv)$ in scalar and vectorial cases. In the second section, description of the Difference quotient method followed by a look into the regularity of singular set of the minimizer. Third section contains few important theorems regarding the regularity and type of the minimizer. In fourth section we learn about the fact that the regularity of the minimizer does not deteriorate with the increase in the dimension of the domain. Fifth section has been devoted to the partial regularity of the minimizers for more general variational integrals through Variational difference quotient method. The last section is all about quasiconvexity of the integrand, uniform porosity and consequences.

1. Section

Background and Hilbert´s 19th problem: The Regular Variational Integral, actually known as Hilbert´s 19th Problem formed the background of the discussion. It can be defined in the following way:

$$F´[u,\Omega]:=\int_\Omega F(Du(X))\,dX$$

is a regular variational iff $F:\mathbb{R}^{Nn}\to\mathbb{R}$ is $C^2$and there exist constants $0<l\leq L<\infty$ such that

$$l|\tilde{z}|^2\leq F´´(z)[z,\tilde{\tilde{z}}]\leq L|\tilde{z}|^2\;\forall z,\tilde{z}\in\mathbb{R}^{Nn}$$

The above condition is referred to as the Hypothesis **(H)** of the problem. In this course the intention was to generalize Weyl´s lemma to the minimizers of the Regular variational integral.

Lemma 1. ***(Weyl)****: Minimizers of Dirac integral are C^{∞}- smooth.*

Definition 2. (Dirac integral): $Dir(v, \Omega) = \int_{\Omega} |Dv|^2$

Problem under consideration: The problem under consideration is indicated by **(P)**, which is as follows:

(P): Given a bounded, open subset $\Omega \subset \mathbb{R}^n$ and $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, we have to find $u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$ such that

$\int_{\Omega} F(Du) \leq \int_{\Omega} F(Dv)\ \forall v \in W_g^{1,2}(\Omega, \mathbb{R}^N)$.

(P) is an example of a regular variational problem. The solutions to **(P)** are called F- minimizers.

Existence and Uniqueness of solution: It has been proved that solutions to **(P)** exist and they are unique. The existence is proved by direct method and uniqueness is implied by the strict convexity of F.

Regularity of solution:

Case 1. If $N = 1$, i.e., scalar case, then the answer is ´Yes´. Based on the work of DeGiorgi- Nash - Moser, Lady $\tilde{Zh}$enskaya and Ural´seva showed that $u \in C_{loc}^{1,\alpha}(\Omega)\ \forall \alpha < 1$.

Case 2. If $N > 1$,i.e. vectorial case, then the answer is ´No´, in general. Minimizers can have singularities (not continuous, unbounded, not differentiable, etc.).

Counter-examples for regularity of solution in vectorial case:

- Neĉas(1975): There exists regular variational problem such that
$$u = u(X) : B(0,1) \subset \mathbb{R}^n \to \mathbb{R}^{n^2}$$
where
$$u^{ij}(X) = \frac{X^i X^j}{|X|}$$
is a minimizer when $n \geq 25$. Here $|X|$ is a symmetric matrix. The solution u is not differentiable but Lipschitz in very high dimensions.
- Hao, Leonardi & Neĉas (1996): It is similar as above but with
$$u^{ij}(X) = \frac{X^i X^j}{|X|} - \frac{|X|}{n}\delta_{ij}\ n \geq 5$$
- $\tilde{S}$verak, Yan(2000,2002): Same as above but with
$$u^{ij}(X) = |X|^{-\varepsilon(n)}\left(\frac{X^i X^j}{|X|} - \frac{|X|}{n}\delta_{ij}\right)$$
such that $\varepsilon(n) > 0$ and $u \in W_{loc}^{1,\infty}$when $n \geq 3$ and $u \notin L_{loc}^{\infty}$when $n \geq 5$. Hence regularity worsens as n increases.

Now we introduce the concept of ´Partial regularity´ and few important remarks regarding it.

Definition 3. (Partial Regularity): It is defined as the regularity of some ´small´ relatively closed set.

Theorem. *(Morrey, Giusti, Miranda et al)(1968): Let $u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$ be the minimizer of* **(P)** *and satisfies* **(H)***. Then*

$$\Omega_u := \{X \in \Omega : \underset{r\to 0}{\underline{Lt}} \fint_{B(X,r)} |Du - (Du)_{X,r}|^2 = 0 \ \& \ \underset{r\to 0}{\overline{Lt}} |(Du)_{X,r}| < \infty\}$$

is open, regular and u is $C^{1,\alpha}_{loc}(\Omega_u)\ \forall \alpha < 1$.

Remark. By Lebesgue´s differentiation theorem the singular set Σ_u, where $\Omega \backslash \Omega_u = \Sigma_u$ has Lebesgue measure zero.

But we can estimate the size of Σ_u much more efficiently if we use $u \in W^{2,2}_{loc}$ and Hausdorff measures.

2. Section

Difference-Quotient Method:(Nirenberg - Shiffmann)(1950s). It states that *$u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$ satisfies* **(B)** *and solves* **(P)** *then $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ and for any ball $B(X, 3R) \subset \Omega$ u satisfies the relation*

$$\int_{B_R} |\triangle_h Du|^2 \leqq (\frac{2L}{l})^2 R^2 h^2 \int_{B_{3R}} |Du - Da|^2$$

for all affine maps $a : \mathbb{R}^n \to \mathbb{R}^N$ and all increments $|h| < R$ where $h \in \mathbb{R}^n$.

Notation: Let $f : \Omega \to \mathbb{R}^k$, and $h \in \Re^n$. Define

$$\triangle_h f(X) = f(X + h) - f(X),\ X, X + h \in \Omega$$

Proof. As u is F- minimizing hence u satisfies the Euler-Lagrange equation:

$$\int_\Omega F´(Du)[D\phi] = 0\ \forall \phi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \tag{2.1}$$

We fix a ball $B(X, 3R) \subset \Omega$ and take a Lipschitz cut-off function $\rho \in W_0^{1,\infty}(B_{3R})$ such that $\chi_{B_R} \leqq \rho \leqq \chi_{B_{3R}}$ and $|D\rho| \leqq \frac{1}{R}$ *a.e.* We put $\phi = \triangle_{-h}(\rho^2 \triangle_h(u - a))$, for $|h| < R$. Then ϕ is an admissible test map in (2.1). Hence $\phi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$. Hence

$$\begin{aligned} 0 &= \int_\Omega F´(Du)D[\triangle_{-h}(\rho^2 \triangle_h(u - a))] \\ &= \int_\Omega \triangle_h F´(Du)[\triangle_h(u - a)D\rho^2 + \rho^2 \triangle_h D(u - a)] \end{aligned}$$

□

Now,

$$\triangle_h F´(Du)\triangle_h Du = \int_0^1 F´´(Du + t\triangle_h Du)[\triangle_h Du, \triangle_h Du]\, dt$$

By **(H)**

$$l|\triangle_h Du|^2 \leqq \int_0^1 F´´(Du + t\triangle_h Du)[\triangle_h Du, \triangle_h Du]\, dt \leqq L|\triangle_h Du|^2$$

$$\therefore l \int_\Omega \rho^2 |\triangle_h Du|^2 \leqq \int_\Omega \rho^2 \triangle_h F´(Du)\triangle_h Du \leqq L \int_\Omega \rho^2 |\triangle_h Du|^2$$

Now,

$$
\begin{aligned}
\int_\Omega \rho^2 |\triangle_h D(u-a)|^2 &= \int_\Omega \rho^2 \triangle_h D(u-a)\overline{\triangle_h D(u-a)} \\
&= -\int_\Omega D(\rho^2 \triangle_h D(u-a))\overline{\triangle_h (u-a)} \\
&= -\int_\Omega D(\rho^2)\triangle_h D(u-a)\overline{\triangle_h (u-a)} - \int_\Omega \rho^2 D(\triangle_h D(u-a))\overline{\triangle_h (u-a)} \\
\Rightarrow \int_\Omega \rho^2 |\triangle_h D(u-a)|^2 &+ \int_\Omega \rho^2 D(\triangle_h D(u-a))\overline{\triangle_h (u-a)} \\
&= -\int_\Omega D(\rho^2)\triangle_h Du\overline{\triangle_h (u-a)} \\
&\leqq |\int_\Omega D(\rho^2)\triangle_h Du\overline{\triangle_h (u-a)}| \\
\Rightarrow \int_\Omega \rho^2 |\triangle_h D(u-a)|^2 &\leqq \int_\Omega |D(\rho^2)||\triangle_h Du||\triangle_h (u-a)|
\end{aligned}
$$

$$
\therefore l\int_\Omega \rho^2|\triangle_h Du|^2 - L\int_\Omega |\triangle_h Du||\triangle_h(u-a)||D(\rho^2)| \leqq 0 \tag{2.2}
$$

From (2.2) we get

$$
\begin{aligned}
\int_\Omega \rho^2 |\triangle_h Du|^2 &\leqq \frac{L}{l}\int_\Omega |\triangle_h Du||\triangle_h(u-a)||D(\rho^2)| \\
&= \frac{2L}{l}\int_\Omega \rho|\triangle_h Du||\triangle_h(u-a)||D\rho|
\end{aligned}
$$

Applying Cauchy- Shwartz inequality to the above relation we get.

$$
\begin{aligned}
\int_\Omega \rho^2|\triangle_h Du|^2 &\leqq \frac{2L}{l}(\int_\Omega \rho^2|\triangle_h Du|^2)^{1\backslash 2}(\int_\Omega |\triangle_h(u-a)^2|D\rho|^2)^{1\backslash 2} \\
&\leqq (\frac{2L}{l})^2\int_\Omega |\triangle_h(u-a)|^2|D\rho|^2
\end{aligned} \tag{2.3}
$$

Applying the properties of the cut-off function ρ to (2.3) we obtain

$$
\int_{B_R} |\triangle_h Du|^2 \leqq (\frac{2L}{l})^2 R^2 \int_{B_{2R}} |\triangle_h(u-a)|^2 \tag{2.4}
$$

Now,

$$
\begin{aligned}
\int_{B_{2R}} |\triangle_h(u-a)|^2 &= \int_{B_{2R}} |\int_0^1 D(u-a)(X+th)h\,dt|^2 \\
&= \int_{B_{2R}} |\int_0^1 D(u-a)(X+th)\,dt|^2 h^2 \\
&\leqq \int_{B_{2R}} \int_0^1 |D(u-a)(X+th)|^2\,dt\,dX\,h^2 \\
&= \int_0^1 \int_{B_{2R}} |D(u-a)(X+th)|^2\,dX\,dt\,h^2 \\
&= \int_0^1 \int_{B_{2R+th}} |D(u-a)|^2\,dX\,dt\,h^2 \\
&\leqq \int_{B_{3R}} |D(u-a)|^2 h^2 \qquad (2.5)
\end{aligned}
$$

Hence, combining (2.4) and (2.5) we get

$$
\begin{aligned}
&\int_{B_R} |\triangle_h Du|^2 \leqq (\frac{2L}{l})^2 R^2 h^2 \int_{B_{3R}} |Du-Da|^2 \\
&\Rightarrow \int_{B_R} |D^2u|^2 \leqq (\frac{2L}{l})^2 R^2 \int_{B_{3R}} |Du-Da|^2 \qquad (2.6)
\end{aligned}
$$

Hence $Du \in W^{1,2}_{loc} \Rightarrow u \in W^{2,2}_{loc}$.

Putting $Da = (Du)_{X,3R}$ in (2.6) we get

$$
\int_{B_R} |D^2u|^2 \leqq (\frac{2L}{l})^2 R^2 \int_{B_{3R}} |Du-(Du)_{X,3R}|^2 \qquad (2.7)
$$

(2.7) is known as the *Caccioppoli inequality of 1st kind.*
Furthermore, for each $1 \leqq s \leqq n$

$$
\int_\Omega F''(Du)[D(D_su), D\phi] = 0,\ \phi \in W_0^{1,2}(\Omega, \mathbb{R}^N) \text{and } dist(spt\phi, \partial\Omega) > 0
$$

. This can be proved in the following way.

We put $\phi = D_s\psi$, where $\psi \in C_c^\infty(\Omega, \mathbb{R}^N)$

Hence putting the value of ϕ in (2.1) we get

$$
\int_\Omega F'(Du).DD_s\psi\,dX = 0 \Rightarrow \int_\Omega F''(Du)[D_sDu, D\psi]\,dX = 0 \qquad (2.8)
$$

$\because D(D_su) \in L^2(\Omega, \mathbb{R}^{N*n})$ and $C_c^\infty(\Omega, \mathbb{R}^N)$ is dense in $W_0^{1,2}(\Omega, \mathbb{R}^N)$, hence (2.8) becomes

$$
\int_\Omega F''(Du)[DD_su, D\phi]\,dX = 0 \qquad (2.9)
$$

Lemma 4. *(Gehring): Let Ω be an open subset of $\mathbb{R}^n$ and let $f \in L^1_{loc}(\Omega)$ be a non negative function such that*

$$
\fint_B f^p\,dX \leqq b_1(\fint_{2B} f\,dX)^p + b_2
$$

for some constants $b_1, b_2 > 0\,, p > 1$ and for any ball $B(2B \subseteq \Omega)$. Then there exists $s > 1$ and $c > 0$ so that

$$(\fint_B f^{p^s}\, dX)^{1/(ps)} \leqq c(1 + \fint_{2B} f\, dX)$$

Now, *Poincare´- Sobolev inequality, Gehring´s lemma* and (2.7) yields that: $\exists \varepsilon = \varepsilon(n, \frac{L}{l}) > 0$ *such that* $u \in W^{2,2+\varepsilon}_{loc}(\Omega, \mathbb{R}^N)$.

$$\therefore \begin{cases} \Sigma_n = \emptyset & when\, n = 2 \\ dim_H \Sigma_n < n - 2 & when\, n > 2 \end{cases}$$

$u \in C^{0,\mu}_{loc}(\Omega, \mathbb{R}^N)$ for some $\mu > 0$ when $n \leq 4$.

3. Section

Theorem 5. *(G. R. Mingione & J. Kristensen): Let $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ solves* **(P)**. *Fix Hölder´s exponent:* $0 < \alpha \leq 1$.

Then

(1) $u \in C^{1,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ $\quad$ *if* $\frac{L}{l} \leq \frac{n-1}{n-2+2\alpha}$

(2) $u \in C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ $\quad$ *if* $\frac{L}{l} \leq \frac{n-1}{n-4+2\alpha}$ *or* $n - 4 + 2\alpha \leq 0$.

In particular u is Hölder continuous if

$\frac{L}{l} < 4 \quad when\, n = 5$

or

$\frac{L}{l} < \frac{5}{2} \quad when\, n = 6$.

Definition 6. (*Q-minimizer*): Let $Q \geq 1$. Then $w \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a Q-quasiminimizer, or shortly, Q-minimizer for $F[., \Omega]$ iff

$$F[w, \Omega´] \leq Q\, F[w + \phi, \Omega´]$$

for all $\phi \in W^{1,2}_0(\Omega´, \mathbb{R}^N)$ and all open subsets $\Omega´ \subset \Omega$.

Theorem 7. *(Ziemer): Let $Q \geq 1$. Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a spherical Q-minimizer for Dirichlet integral iff for any open ball $B \subset \Omega$*

$$Dir(u, B) \leq Q\, Dir(u + \phi, B)\, \forall \phi \in W^{1,2}_0(B, \mathbb{R}^N)$$

Then the map

$$r \mapsto r^{-\frac{n-1}{Q}} \int_{B_r(X)} |Du|^2$$

is non-decreasing for $r \in (0, dist(X, \partial\Omega))$.

Note: We fix an open ball $B \subset \Omega$ and let $h \in W^{1,2}_{D_s u}(\Omega´, \mathbb{R}^N)$ be harmonic.
Then $\phi = h - D_s u$ is admissible in (2.9). Hence we get

$$\begin{aligned} l \int_B |DD_s u|^2 &\leq \int_B F^{\cdot\cdot}(Du)[DD_s u, DD_s u] \\ &= \int_B F^{\cdot\cdot}(Du)[DD_s u, Dh] \\ &\leq \int_B F^{\cdot\cdot}(Du)[Dh, Dh] \\ &\leq L \int_B |Dh|^2 \end{aligned}$$

$$\Rightarrow Dir[D_s u, B] \leq \frac{L}{l} Dir[h, B]$$

$\therefore$ $D_s u$ is a Q-minimizer for $Dir[.,\Omega]$ when $Q=\frac{L}{l}$.

Proof. (*J. Kristensen & G. R. Mingione*)
Let H_k= {harmonic, homogeneous polynomials of degree k on $\mathbb{R}^N$}
$B=B(0,1)$ and $S=\partial B(0,1)$
$H_k(S)=\{p|_S : p\in H_k\}$. □

Fact. *1) Define $L^2(S)=\oplus_{k=0}^{\infty}H_k(S)$. Then $L^2(S)$ is a Hilbert space.*
If $p\in H_k$, $q\in H_l$,where $k\neq l$, then inner product of p and q is zero.
2) If $p,q\in H_k$, then $\int_S Dp.Dq = k(n+2k-2)\int_S pq$.

Now, $B_r=B(X,r)$. Without loss of generality, we assume $X=0$.
Given: $u\in W^{1,2}_{loc}(\Omega,\mathbb{R}^N)$ is a spherical Q-minimizer for $Dir[.,\Omega]$, i.e.,

$$\int_{B_r}|Du|^2\le Q\int_{B_r}|Dv|^2$$

where $v\in W^{1,2}_u(B_r,\mathbb{R}^N)$ is harmonic.
Observation:

- $u|_{\partial B_r}\in W^{1,2}(\partial B_r,\mathbb{R}^N)$ and tangential derivative $|D_t(u|_{\partial B_r})|\le|Du|$ a.e. on ∂B_r.
- We put $D(r)=\int_{B_r}|Du|^2$.

Note: $D(r)\ge 0$ is non-decreasing and absolutely continuous with $D'(r)=\int_{\partial B_r}|Du|^2$a.e. r.
For $0<r<dist(X=0,\partial\Omega)$ we fix $r=R$ satisfying the above two conditions.
Define $v(x)=\frac{1}{R}u(Rx)$, $x\in B_r$.

$$\therefore \int_B |Du|^2 = R^n\int_B |Du(Rx)|^2\,dx$$
$$=R^n\int_B |Dv|^2\,dx$$

Calculation:
Let $h\in W^{1,2}_v(B_r)$ be harmonic. $v|_S=\sum_{k=0}^{\infty}p_k$, $p_k\in H_k$. $\because h=v|_{s=\partial B_r}$,
$\therefore h=\sum_{k=0}^{\infty}p_k$ and $Dh=\sum_{k=1}^{\infty}Dp_k$ in $L^2(B_r)$.

$$\because \int_{B_r}|Dv|^2\le Q\int_{B_r}|Dh|^2$$
$$=Q\sum_{k=1}^{\infty}\int_{B_r}|Dp_k|^2 .$$
$$=Q\sum_{k=1}^{\infty}\frac{1}{n+2k-2}\int_S|Dp_k|^2$$

Now, $|Dp_k|^2=|D_t p_k|^2+|D_\nu p_k|^2$ by Pythagoras´ theorem, where $D_\nu p_k=\frac{\partial p_k}{\partial\nu}\otimes\nu = kp_k\otimes\nu$. Hence,

$$\int_S |Dp_k|^2 = \int_S [|D_t p_k|^2 + |D_\nu p_k|^2]$$
$$= \int_S [|D_t p_k|^2 + k^2 |p_k|^2]$$
$$= \int_S [|D_t p_k|^2 + \frac{k}{n+2k-2} |Dp_k|^2]$$
$$\Rightarrow \frac{n+k-2}{n+2k-2} \int_S |Dp_k|^2 = \int_S |D_t p_k|^2$$
$$\Rightarrow \int_S |Dp_k|^2 = \frac{n+2k-2}{n+k-2} \int_S |D_t p_k|^2$$
$$\Rightarrow \int_{B_r} |Dp_k|^2 = \frac{1}{n+k-2} \int_S |D_t p_k|^2$$
$$\therefore \int_{B_r} |Dv|^2 \le Q \sum_{k=1}^{\infty} \frac{1}{n+k-2} \int_S |D_t p_k|^2$$
$$\le \frac{Q}{n-1} \sum_{k=1}^{\infty} \int_S |D_t p_k|^2$$
$$\overset{(*)}{=} \frac{Q}{n-1} \int_S |D_t v|^2$$
$$\le \frac{Q}{n-1} \int_S |Dv|^2$$

(*)For $k \neq l$, $\int_S D_t p_k . D_t p_l = -kl \int_S p_k . p_l = 0$.

$$\therefore D(r) = \int_{B_r} |Du|^2 \le \frac{Qr^n}{(n-1)r^{n-1}} \int_S |Dv|^2$$
$$= \frac{Qr}{n-1} \int_S |Du|^2$$
$$= \frac{Qr}{n-1} D'(r) \; for \; a.e. \; r \in (0, dist(X=0, \partial\Omega))$$

$\therefore$ for almost every r with $D(r) > 0$,

$$\frac{D'(r)}{D(r)} \ge \frac{n-1}{Qr}$$

and so for $0 < r < R < dist(0, \partial\Omega)$ we get by FTC,

$$\frac{n-1}{Q} . log \frac{R}{r} \le log \frac{D(R)}{D(r)}$$
$$\Rightarrow r^{-\frac{n-1}{Q}} D(r) \le R^{-\frac{n-1}{Q}} D(R)$$

$\Rightarrow r \mapsto r^{-\frac{n-1}{Q}} \int_{B_r(X)} |Du|^2$ is non decreasing and $u(x)$ as well as $v(x) = \frac{1}{R} u(Rx)$ are spherical$Q$-minimizers of $Dir[., \Omega]$.

4. Section

In this section we discuss that with increasing dimension, i.e., $n \to \infty$ regularity of the minimizer does not deteriorate.

Theorem 8. *(J. Kristensen & C. Melcher)(2008): Let $u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$ be a solution of* **(P)**. *Then*

$$u \in W_{loc}^{2,2+\frac{1}{50}\frac{l}{L}} \; \forall n, N.$$

Proof. (Sketch of the proof):

[We know: $u \in W_{loc}^{2,2}(\Omega, \mathbb{R}^N)$ and $0 = \int_\Omega \triangle_h F'(Du)[D\phi]$ for $\phi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ with $dist(spt\,\phi, \partial\Omega) > |h|$].

So we first seek a test map ϕ such that $D\phi \sim |\triangle_h Du|^\varepsilon \triangle_h Du$.

We fix a ball $B \subset\subset \Omega$.

Let $\rho \in C_c^\infty(B)$ be a cut-off function.

We take $\varepsilon > 0$ such that $u \in W_{loc}^{1,2+\varepsilon}(\Omega, \mathbb{R}^N)$, (i.e., ε depends on n) but $w = \rho\triangle_h u \in W^{1,2+\varepsilon}(\mathbb{R}^n)$.

We take $\psi \in W^{1,2+\varepsilon}$ such that $D\psi = E(|Dw|^\varepsilon Dw)$. Here E is the Helmholtz projection onto gradient fields on $\mathbb{R}^n$.

It can be written as $E = -R \otimes R$, where R is the Riesz transforms on $\mathbb{R}^n$.

Now, $D\psi$ can be written as $D\psi = |Dw|^\varepsilon Dw - \sigma$, $div\sigma = 0$.Here σ is the error-term.

From the results of Iwaniec & Martin, Iwaniec & Sbordone

$$||\sigma||_{\frac{2+\varepsilon}{1+\varepsilon}} \leq c\varepsilon ||Dw||_{2+\varepsilon}^{1+\varepsilon},\; c < 50 \tag{4.1}$$

We put $\phi = \rho(\psi - \psi_B)$in the Euler-Lagrange´s equation (2.1).

$\therefore D\phi = \rho D\psi + lower\, order\, terms(l.o.t)$

$= \rho|D(\rho\triangle_h u)|^\varepsilon D(\rho\triangle_h u) - \rho\sigma + l.o.t.$

$\sim \rho^{2+\varepsilon}|\triangle_h Du|^\varepsilon \triangle_h Du - \rho\sigma + l.o.t.$

So, from (2.1) we get

$$\begin{aligned} 0 &= \int_B \triangle_h F'(Du).D\phi \\ &\sim \int_B \triangle_h F'(Du).[\rho^{2+\varepsilon}|\triangle_h Du|^\varepsilon \triangle_h Du - \rho\sigma + l.o.t.] \\ &= \int_B \rho^{2+\varepsilon}\triangle_h F'(Du).\triangle_h Du|\triangle_h Du|^\varepsilon - \int_B \triangle_h F'(Du).(\rho\sigma) + l.o.t. \end{aligned}$$

Now, from **(H)**,

$$l\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon} \leq \int_B \rho^{2+\varepsilon}|\triangle_h Du|^\varepsilon \triangle_h F'(Du).\triangle_h Du \leq L\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon} \tag{4.2}$$

and

$$l\int_B |\triangle_h Du||\rho\sigma| \leq \int_B \triangle_h F'(Du).(\rho\sigma) \leq L|\triangle_h Du||\rho\sigma| \tag{4.3}$$

Hence from (4.2) and (4.3) we get

$$l\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon} - L\int_B \rho|\triangle_h Du||\sigma| + l.o.t. \leq 0$$

$$\Rightarrow \int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon} \leq \frac{L}{l}\int_B \rho|\triangle_h Du||\sigma| + l.o.t.$$

$$\leq \frac{L}{l}(\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon})^{\frac{1}{2+\varepsilon}}||\sigma||_{\frac{2+\varepsilon}{1+\varepsilon}} + l.o.t.(By\,H\ddot{o}lder\acute{\,}s\,inequality)$$

$$\leq \frac{L}{l}c\varepsilon(\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon})^{\frac{1}{2+\varepsilon}}||Dw||^{1+\varepsilon}_{2+\varepsilon} + l.o.t.(By\,(4.1))$$

$$\leq \frac{L}{l}c\varepsilon\int_B \rho^{2+\varepsilon}|\triangle_h Du|^{2+\varepsilon} + l.o.t.(By\,Young\acute{\,}s\,inequality)$$

If $\frac{L}{l}c\varepsilon < 1$, i.e., $\varepsilon < \frac{1}{c}\frac{l}{L}$, then the first term of the right hand side can be absorbed to the left hand side and hence the proof. □

Note: The above theorem is regarding the regularity of the minimizer within the set which forms its domain and not at the boundary of the domain.

5. Section

Here we discuss about the partial regularity of the minimizers for more general variational integrals through Variational difference quotient method.

Variational Difference Quotient Method:(J. Kristensen & G.R.Mingione). Set-up: $\mathcal{F}(v) = \int_\Omega F(X, v, Dv)\,dX$ is a variational integral where Ω is an open, bounded subset of $\mathbb{R}^n$, $v \in W^{1,2}$. We look for minimizer for $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ which satisfies

(H1) $z \mapsto F(X, y, z)$ is C^2 and $(X, y, z) \mapsto F_{zz}(X, y, z)$ is C^0.

(H2) $l|\tilde{z}|^2 \leq F_{zz}(X, y, z)[\tilde{z}, \tilde{z}] \leq L|\tilde{z}|^2$ for all $z, \tilde{z} \in \mathbb{R}^{Nn}$, $X \in \Omega, y \in \mathbb{R}^N$.

(H3) $l|z|^2 - L \leq F(X, y, z) \leq L(|z|^2 + 1)$ for all $X \in \Omega, y \in \mathbb{R}^N, z \in \mathbb{R}^{Nn}$.

(H4) $|F(X_1, y_1, z) - F(X_2, y_2, z)| \leq Lw_\alpha(|X_1 - X_2| + |y_1 - y_2|)(|z|^2 + 1)$ for all $X_1, X_2 \in \Omega, y_1, y_2 \in \mathbb{R}^N, z \in \mathbb{R}^{Nn}$ where $w_\alpha(t) = min\{1, t^\alpha\}$, $, t \geq 0$, $0 < \alpha \leq 1$.

We can find a $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ which is a minimizer for $\mathcal{F}$ and the existence of u follows from by the use of direct method and nothing can be said about its uniqueness since $\mathcal{F}(v)$ is not convex.

Theorem 9. *(Giaquinta & Giusti)(1979): Let $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ be an $\mathcal{F}$ minimizer and satisfies* H(1-4).

Theorem. *Define*

$$\Omega_u = \{X \in \Omega : \underline{lim}_{r\to 0} \fint_{B(X,r)} |Du - (Du)_{(X,r)}|^2 = 0\,\&\,\overline{lim}_{r\to 0}(|u_{(X,r)}| + |(Du)_{(X,r)}|) < \infty\}$$

Then Ω_u is open and $u \in C^{1,\frac{\alpha}{2}}_{loc}(\Omega_u, \mathbb{R}^N)$.

From Lebesgue´s differentiation theorem, $\mathcal{L}^n(\Sigma_u) = 0$ where $\Sigma_u = \Omega \backslash \Omega_u$. Hence, Σ_u is a singular set.

Remark. Assume H(1-4) are satisfied, $\alpha = 1$ and $F(X, y, z) \in C^\infty$. Then an F-minimizer $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is also an F-extremal, i.e., it solves the Euler-Lagrange equation for F:

$$divF_z(X, u, Du) = F_y(X, u, Du)\,in\,\Omega$$

Here $|F_y(X,u,Du)| \leq L(|Du|^2+1)$ by (H4) with $\alpha = 1$. So the above Euler- Lagrange equation is an elliptic system with critical growth ($Du \in L^2$so $F_y(X,u,Du) \in L^1$) and its known that there is no partial regularity theory for weak solutions of such system(e.g. harmonic maps systems, by Rivièra). Hence its important that we are dealing with minimizers and not merely extremals.

Next we try to see whether better size estimates of the singular set Σ_u are possible or not and the answer is yes they are possible if we look into Hausdorff dimension.

Theorem 10. *(2006): Let $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ be F-minimizer and satisfies* H(1-4). *If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ for $p > 2$ then*

$$dim_H \Sigma_u \leq n - min\{\alpha, p-2\}$$

Remark. It follows from Gehring´s lemma that $u \in W^{1,q}_{loc}(\Omega, \mathbb{R}^N)$ for some $q > 2$.

Definition 11. (Sobolev-Slobodetskii space): Let $0 < \theta < 1$. Then $w \in W^{\theta,2}(B)$ iff

$$||w||_{W^{\theta,2}} = ||w||_{L^2} + (\int_B \int_B \frac{|w(x)-w(y)|^2}{|x-y|^{n+2\theta}} dxdy)^{\frac{1}{2}} < \infty$$

The second norm on the RHS is known as Gagliardo semi-norm and is wriiten as $[w]_{\theta,2:B}$.

Note1: $w \in W^{1+\theta,2}(B)$ iff $w \in W^{1,2}$ and $Dw \in W^{\theta,2}$.

Note2: $W^{\theta,2} = B^{\theta,2}_2$, where B is the Besov space.

Theorem. *(G.R.Mingione)(2003): $u \in W^{1+\theta,2}_{loc}(\Omega, \mathbb{R}^N)\ \forall \theta < \frac{\alpha}{2}$.*

Theorem 12. *(2006): Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be F-minimizer and satisfies* H(1-4). *If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ for $p > 2$ then*

$$u \in W^{1+\theta,2}_{loc}(\Omega, \mathbb{R}^N)\ \forall \theta < \frac{1}{2} min\{\alpha, p-2\}.$$

Proof. (Sketch of the proof): We fix a ball $B(X_0,R) \subset\subset \Omega$.

Frozen integrand $\tilde{F}(z) = F(X_0, u_B, z)$.

Let $v \in W^{1,2}_u(B)$ minimize $\int_\Omega \tilde{F}(Dw)$.

From (H2) and Difference quotient method, for cube $Q = Q(X_0, \frac{R}{3\sqrt{n}}) \subset B$

$$\int_Q |\triangle_h Dv|^2 \leq c \frac{|h|^2}{|Q|^{\frac{2}{n}}} \int_{3Q} |Dv - Da|^2$$

where a is affine and $|h| < \frac{1}{2}|Q|^{\frac{1}{n}}$.

From (H2) and the fact $(Dv)_B = (Du)_B$ we get

$$\int_B |Dv - (Dv)_B|^2 \leq \frac{L}{l} \int_B |Du - (Du)_B|^2$$

Estimates:

(1) $\int_Q |\triangle_h Dv|^2 \leq c \frac{|h|^2}{|Q|^{\frac{2}{n}}} \int_B |Du - (Du)_B|^2$.

We put $\sigma = min\{\alpha, p-2\}$.

From minimality condition and (H4) we get

(2) $\int_B |Du - Dv|^2 \leq c \int_B (|Du|^{2+\sigma}+1) R^\sigma$.

For dimensions $(n-2)$ like in (2), $2+\sigma \leq p$ so $|Du| \in L^{2+\sigma}_{loc}$.

Step 1: (For the proof) We have to show $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{Nn})$ $\theta < \frac{\sigma}{2+\sigma}$.
Let $0 < \beta < 1$. Let $B = B(X_0, |h|^\beta) \subset\subset \Omega$.
Combining (1) and (2) we get

$$\int_Q |\triangle_h Du|^2 \leq 2\int_Q |\triangle_h(Du - Dv)|^2 + |\triangle_h Dv|^2$$

$$\overset{|h|\, small}{\leq} 8\int_B |Du - Dv|^2 + 2\int_Q |\triangle_h Dv|^2$$

$$\overset{R=|h|^\beta}{\leq} c\int_B (|Du|^{2+\sigma} + 1)(|h|^{\sigma\beta} + |h|^{2-2\beta}) \; [\because |Q|^{\frac{1}{n}} \sim R]$$

We assume β such that $\sigma\beta = 2 - 2\beta$,i.e.,

$$\int_Q |\triangle_h Du|^2 \leq c\int_B (|Du|^{2+\sigma} + 1)|h|^{\frac{2\sigma}{2+\sigma}}$$

Now we fix $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. For small $|h|$ we cover Ω' by non overlapping cubes $\bar{Q}$ such that $3\sqrt{n}Q \subset \Omega''$.
Note that cubes $3\sqrt{n}Q$ have bounded overlap property $(3^n - 1)(3\sqrt{n} - 1)$.

$$\therefore \int_{\Omega'} |\triangle_h Du|^2 \leq \sum \int_Q |\triangle_h Du|^2 \leq c\int_{\Omega''} (|Du|^{2+\sigma} + 1)|h|^{\frac{2\sigma}{2+\sigma}}.$$

$\therefore Du \in N^{\frac{\sigma}{2+\sigma},2}_{loc}$(Nikolskii space). Now $N^{\frac{\sigma}{2+\sigma},2}_{loc} = B_2^{\frac{\sigma}{2+\sigma},\infty} = W^{\frac{\sigma}{2+\sigma},2}$.
Hence, $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{Nn})$ $\theta < \frac{\sigma}{2+\sigma}$.
Step 2: We have to show that if $0 < \theta < \frac{\sigma}{2}$ and $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{Nn})$, then

$$Du \in W^{t,2}_{loc} \; \forall t < \frac{\sigma}{2(1-\theta) + \sigma}.$$

Note: $\theta < \frac{\sigma}{2(1-\theta)+\sigma}$ iff $\theta < \frac{\sigma}{2}$. Hence this is an improvement.
As in Step 1, but using Poincarè inequality in estimate (1) we get

$$\int_Q |\triangle_h Dv|^2 \leq c\frac{|h|^2}{|Q|^{\frac{2}{n}}}\int_B |Du - (Du)_B|^2$$

$$\overset{Poincar\grave{e}}{\leq} c\frac{|h|^2}{|Q|^{\frac{2}{n}}}|B|^{\frac{2\theta}{n}}[Du]^2_{\theta,2;B}$$

$$\leq c|h|^{2-2\beta+2\theta\beta}[Du]^2_{\theta,2;B}$$

We now use iteration. Define $H(t) = \frac{\sigma}{2(1-t)+\sigma}$, $t \in [0, 1]$.
Note: H is increasing, i.e., $t < H(t) < \frac{\sigma}{2}$ for $0 < t < \frac{\sigma}{2}$ and $H(\frac{\sigma}{2}) = \frac{\sigma}{2}$.
Thus Step 2 reduces to: $Du \in W^{\theta,2}_{loc}$, $0 < \theta < \frac{\sigma}{2}$ this implies $Du \in W^{t,2}_{loc}$, $\forall t < H(\theta)$.
Define two sequences $\{\theta_j\}$ and $\{\gamma_j\}$: $\theta_0 \in (0, \frac{\sigma}{2+\sigma})$ and $\gamma_0 = \frac{\theta_0}{2} + \frac{\sigma}{2(2+\sigma)}$.
We put $\theta_j = H(\theta_{j-1})$ and $\gamma_j = \frac{\theta_j + H(\gamma_{j-1})}{2}$.
Then $0 < \theta_j < \frac{\sigma}{2}$ and $\theta_j \nearrow \frac{\sigma}{2}$ and
$\theta_j < \gamma_j < \frac{\sigma}{2}$ and $\gamma_j \nearrow \frac{\sigma}{2}$.
Claim: $Du \in W^{\gamma_j,2}_{loc}$ $\forall j$.
Induction: Step 1 yields $\theta = \gamma_0$,i.e., $j = 0$.
Let it be true for $j = k$. Then by Step 2, $Du \in W^{t,2}_{loc}$ $\forall t < H(\gamma_k)$.
For $j = k + 1$,

$$\begin{aligned}\gamma_{k+1} &= \frac{\theta_{k+1} + H(\gamma_k)}{2} \\ &= \frac{H(\theta_k) + H(\gamma_k)}{2} \\ &< H(\gamma_k) \ \because H \, is \, increasing\end{aligned}$$

$\therefore$ It is true for $j = k+1$ also.
Thus our claim is correct.
Combining the Step 1 and 2, the proof is done. □

6. Section

In this section we discuss about Quasiconvexity. It is said to be 'Natural condition´ when $n, N > 1$.

Definition 13. The integrand $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is said to be quasiconvex provided

$$\int_{(0,1)^n} [F(X_0, y_0, z_0 + D\varphi(X)) - F(X_0, y_0, z_0)] \, dX \geq 0$$

for every$\varphi \in C_c^\infty((0,1)^n, \mathbb{R}^N)$, for all $X_0 \in \Omega$, $y_0 \in \mathbb{R}^N$ and $z_0 \in \mathbb{R}^{Nn}$.

This condition replaces the usual convexity condition in multi-dimensional calculus of variations and is essentially equivalent to sequential lower semi-continuity in the weak topology of appropraite Sobolev spaces. As such it is intimately related to the minimizers of $\int_\Omega F(X, v, Dv)$ on Dirichlet classes of Sobolev maps and as such quasiconvexity in a strict form allows to prove the partial regularity of minimizers too.

We try to answer the question regarding $\int_\Omega F(Dv)$ when $F : \mathbb{R}^{Nn} \to \mathbb{R}$ is C^2, $|F^{\prime\prime}(z)| \leq L$ and $F - l|.|^2$ is quasiconvex.

Then it has been proved that

- There exists a minimizer $u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$.
- The minimizer is not unique.(*Spadaro,2009*)
- There is partial regularity of the minimizer.(*Evans 1986, Acerbi & Fusco*)

The key point in the proof of regularity of the minimizer is the *Caccioppoli inequality of* 2^{nd}*kind:*

For $B(X_0, 2R) \subset \Omega$,

$$\int_{B(X_0,R)} |Du - (Du)_{X_0,R}|^2 \leq \frac{c}{R^2} \int_{B(X_0,2R)} |u - a|^2$$

for all affine maps, $a : \mathbb{R}^n \to \mathbb{R}^N$.

Definition 14. (Quadratic excess functional): $E(X_0, R)$ is the quadratic excess functional is defined as

$$E(X_0, R) = \fint_{B(X_0,R)} |Du - (Du)_{X_0,R}|^2$$

Relation between Gagliardo norm and Quadratic excess functional: Let $w \in W^{\theta,2}(\mathbb{R}^n)$, $0 < \theta < 1$.

Then

$$
\begin{aligned}
{}^{2}_{\theta,2} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2\theta}}\,dxdy \\
&\sim \int_{\mathbb{R}^n}\int_0^\infty E(x,r)\frac{1}{r^{1+2\theta}}\,drdx
\end{aligned}
$$

If $w \in W^{1,2}(B(X_0,2R))$, then

$$
\int_{B(X_0,R)}\int_0^R \inf_a \fint_{B(X,r)} |w-a|^2 \frac{1}{r^3}\,drdX \le c\int_{B(X_0,2R)} |Dw|^2
$$

Note: $E(X,r) \le \frac{c}{r^2}\inf_a \fint_{B(X,2r)} |u-a|^2$

Applying to the minimizer u, where $B(X_0,4R) \subset \Omega$

$$
\begin{aligned}
\int_{B(X_0,R)}\int_0^R E(X,r)\frac{1}{r}\,drdX &\le c\int_{B(X_0,R)}\int_0^R \inf_a \fint_{B(X,2r)} |u-a|^2\frac{1}{r^3}drdX \\
&\le \tilde{c}\int_{B(X_0,4R)} |Du|^2
\end{aligned}
$$

Definition 15. (Uniformly δ-porous): $\forall B(X_0,R) \subset \Omega$ there exists $X \in B(X_0,R)$ such that $B(X,\delta R) \subset B(X_0,R) \setminus \Sigma_u$.

Theorem 16. *(2007): If $u \in W^{1,\infty}$, i.e., u is Lipschitz, then Σ_u is uniformly δ-porous for some $0 < \delta < \frac{1}{2}$.*

Consequence: $dim\Sigma_u < n$.

$u \in W^{1,\infty} \Rightarrow \fint_{B(X_0,R)}\int_0^R E(X,r)\frac{1}{r}\,drdX \le C$.

When $u \in W^{1,\infty}$, there exists $\varepsilon > 0$ such that

$$
\Omega_u = \{x \in \Omega : r \in (0, dist(x,\partial\Omega))\, such\, that\, E(x,r) < \varepsilon\}
$$

is a regular set.

Note: If $E(x,r) < 2^{-n}\varepsilon$, then $B(x,\frac{r}{2}) \subset \Omega_u$.

Now we fix a ball $B_R = B(X_0,R) \subset \Omega$ and let $0 < \mu < 1$.

Define

$E = \{x \in B(X_0,R) : E(x,r) \ge 2^{-n}\varepsilon\, \forall r \in (\mu R, R)\}$.

$C \ge \frac{1}{|B_R|}\int_E \int_{\mu R}^R 2^{-n}\varepsilon\frac{1}{r}\,drdx = \frac{|E|}{|B_R|}2^{-n}\varepsilon ln\frac{1}{\mu}$

Thus

$$
\frac{|E|}{|B(X_0,\frac{R}{2})|} \le \frac{2C^{2n}}{E}.\frac{1}{ln\frac{1}{\mu}} \longrightarrow 0\, as\, \mu \longrightarrow 0
$$

If therefore we take

$$
\mu := exp(-\frac{2^{2n+1}C}{\varepsilon})
$$

then $\frac{|E|}{|B(X_0,\frac{R}{2})|} \le \frac{1}{2}$ and it follows that we can find $x \in B(X_0,R/2)\setminus E$ and hence $x \in B(X_0,R) \setminus E$.

Thus, in particular, $B_R \setminus E \neq \emptyset$.

References

[1] Kristensen, Jan;Melcher, Christof. Regularity in oscillatory nonlinear elliptic systems. . Math. Z. 260 (2008), no. 4, 813–847.

[2] Kristensen, Jan; Mingione, Giuseppe The singular set of Lipschitzian minima of multiple integrals. Arch. Ration. Mech. Anal. 184 (2007), no. 2, 341–369.

[3] Kristensen, Jan; Mingione, Giuseppe The singular set of minima of integral functionals. Arch. Ration. Mech. Anal. 180 (2006), no. 3, 331–398.