A stability criterion for two-fluid interfaces

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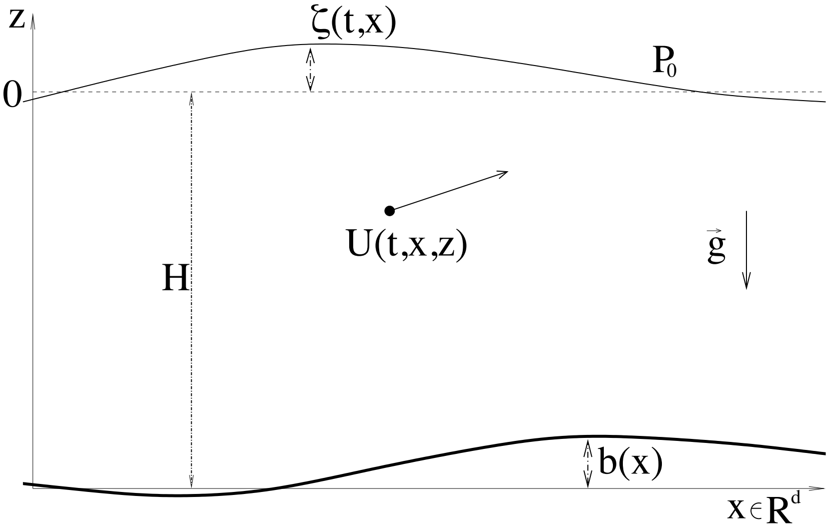
Outline

I- General Introduction

1) The one fluid case (Water Waves)

a) Notations



Equations inside the fluid domain

- *Incompressibility*: $\boxed{\mathrm{div}_{X,z} U = 0}$
- *Irrotationality*: $\boxed{\mathrm{curl}_{X,z} U = 0}$
- *Conservation of momentum (Euler equation)*:

$$\boxed{\rho\big(\partial_t U + (U \cdot \nabla_{X,z})U\big) = -\nabla_{X,z} P + \rho \mathbf{g}}$$

Equations at the surface

- Continuity of the stess tensor: $\boxed{P - P_{atm} = \sigma\kappa(\zeta)}$ $(\sigma \geq 0)$
- Bounding surface: $\boxed{\partial_t \zeta - \sqrt{1 + |\nabla\zeta|^2} U_n = 0}$

Equations at the rigid bottom

- Impermeability: $\boxed{U_n = 0}$

c) Free surface Bernoulli equations

Equations inside the fluid domain

- *Irrotationality*: $\boxed{U = \nabla_{X,z}\Phi}$
- *Incompressibility*: $\boxed{\Delta_{X,z}\Phi = 0}$
- *Bernoulli equation*:

$$\boxed{\rho\big(\partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2\big) = -(P - P_{atm}) - \rho g z}$$

Equations at the surface

- Continuity of the stess tensor: $\boxed{P - P_{atm} = \sigma k(\zeta)}$ $(\sigma \geq 0)$
- Bounding surface: $\boxed{\partial_t\zeta - \sqrt{1 + |\nabla\zeta|^2}\partial_n\Phi = 0}$

Equations at the rigid bottom

- Impermeability: $\boxed{\partial_n\Phi = 0}$

d) The Zakharov-Craig-Sulem formulation

Idea

Write the equations in terms of ζ and $\boxed{\psi(t,x) = \Phi(t,X,\zeta(t,X))}$

- $\partial_t \Phi_{|_{z=\zeta}} = \partial_t \psi - (\partial_t \zeta)\partial_z \Phi_{|_{z=\zeta}}$
- $\nabla \Phi_{|_{z=\zeta}} = \nabla \psi - (\nabla \zeta)\partial_z \Phi_{|_{z=\zeta}} =: \underline{V}$
- $\partial_z \Phi_{|_{z=\zeta}} = \frac{G[\zeta]\psi + \nabla\zeta\cdot\nabla\psi}{1+|\nabla\zeta|^2} =: \underline{w}$

where

$$G[\zeta]\psi := \sqrt{1+|\nabla\zeta|^2}\partial_n \Phi_{|_{z=\zeta}}$$

The operator $G[\zeta]$ is called Dirichlet-Neumann operator

Since $P_{|_{z=\zeta}} = P_{atm} + \sigma\kappa(\zeta)$, the pression disappears when we take the trace at the surface of the Bernoulli equation

$$\rho\big(\partial_t \Phi + \tfrac{1}{2}|\nabla_{X,z}\Phi|^2\big) = -(P - P_{atm}) - \rho g z.$$

After some computations, we obtain therefore

$$(1)\ \begin{cases} \partial_t \zeta - G[\zeta]\psi = 0, \\ \partial_t \psi + g\zeta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{(G[\zeta]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)^2} = -\dfrac{\sigma}{\rho}\kappa(\zeta), \end{cases}$$

where we recall that

- $G[\zeta]\psi = \sqrt{1+|\nabla\zeta|^2}\,\partial_n \Phi_{|_{z=\zeta}}$
- $\kappa(\zeta) = -\nabla\cdot\big(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\big)$

Proposition

If $\zeta = 0$ *then* $G[0]\psi = |D|\tanh(H|D|)$.

Notation: $\widehat{f(D)u} = f(\xi)\widehat{u}(\xi)$.

The linearization of (1) around the rest state $(\zeta = 0, \psi = 0)$ gives

$$\begin{cases} \partial_t\zeta - |D|\tanh(H|D|)\psi = 0, \\ \partial_t\psi + g\zeta = 0 \end{cases}$$

and therefore

$$(2) \qquad \partial_t^2\zeta + g|D|\tanh(H|D|)\zeta = 0.$$

Let us investigate the behavior of the linearized equations (2),

$$\partial_t^2 \zeta + g|D| \tanh(H|D|)\zeta = 0.$$

The deep water limit $H = \infty$

Since $\lim_{x \to +\infty} \tanh(x) = 1$, we replace $\tanh(H|D|)$ by 1 in (2),

$$\partial_t^2 \zeta + g|D|\zeta = 0.$$

This is a dispersive equation.

The shallow water limit $H \ll 1$

Since $\tanh(x) \sim_{x \to 0} x$, we replace $\tanh(H|D|)$ by $H|D|$ in (2),

$$\partial_t^2 \zeta - gH\Delta\zeta = 0.$$

This is a non dispersive equation (in physical terminology !!!).

- *The deep water limit coincides with the symbolic analysis*
- *The SW limit is not compatible with symbolic analysis*

f) One more formulation

Let us denote by $\underline{U}$ the velocity at the surface,

$$\underline{U} = (\underline{V}, \underline{w}) = U_{|_{z=\zeta}}.$$

We take the trace of the Euler equation at the surface

$$\rho\big(\partial_t U + (U\cdot\nabla_{X,z})U\big)_{|_{z=\zeta}} = -\nabla_{X,z}P_{|_{z=\zeta}} + \rho\mathbf{g}$$

- $\partial_t U_{|_{z=\zeta}} = \partial_t \underline{U} \boxed{-(\partial_t\zeta)\partial_z U_{|_{z=\zeta}}}$
- $(U\cdot\nabla_{X,z})U_{|_{z=\zeta}} = (\underline{V}\cdot\nabla)\underline{U} \boxed{-(\underline{V}\cdot\nabla\zeta)\partial_z U_{|_{z=\zeta}} + \underline{w}\partial_z U_{|_{z=\zeta}}}$
- $-\nabla_{X,z}P_{|_{z=\zeta}} + \rho\mathbf{g} = \begin{pmatrix} -\nabla(\underline{P} - P_{atm}) + \nabla\zeta\partial_z P_{|_{z=\zeta}} \\ -\rho g - \partial_z P_{|_{z=\zeta}} \end{pmatrix}$

We get therefore (take $\sigma = 0$ here)

$$\rho\big(\partial_t \underline{U} + (\underline{V}\cdot\nabla)\underline{U}\big) = \begin{pmatrix} \nabla\zeta\partial_z P_{|_{z=\zeta}} \\ -\rho g - \partial_z P_{|_{z=\zeta}} \end{pmatrix},$$

which, together with the kinematic boundary condition, yields

$$(3) \qquad \begin{cases} \partial_t\zeta + \underline{V}\cdot\nabla\zeta - \underline{w} = 0, \\ \partial_t\underline{V} + (\underline{V}\cdot\nabla)\underline{V} + \mathfrak{a}\nabla\zeta = 0, \end{cases}$$

with

$$\mathfrak{a} = -\frac{1}{\rho}\partial_z P_{|_{z=\zeta}} = g + (\partial_t + \underline{V}\cdot\nabla)\underline{w}.$$

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - \underline{w} = 0, \\ \partial_t \underline{V} + (\underline{V} \cdot \nabla) \underline{V} + \mathfrak{a} \nabla \zeta = 0, \end{cases}$$

One must of course express $\underline{w}$ and $\mathfrak{a}$ in terms of ζ and $\underline{V}$.
For the moment, let us proceed formally:

- When $\zeta = 0$ then

$$\begin{aligned} \underline{w} &= \partial_z \Phi_{|_{z=0}} = \partial_n \Phi_{|_{z=0}} \\ &= G[0]\psi = |D| \tanh(H|D|)\psi \\ &= -\tanh(H|D|)\frac{\nabla}{|D|} \cdot \nabla\psi \\ &= -\tanh(H|D|)\frac{\nabla}{|D|} \cdot \underline{V}. \end{aligned}$$

- When $\zeta \neq 0$ it remains true when $d = 1$ (and "almost" true when $d = 2$) that

$$\underline{w} = -\frac{\nabla}{|D|} \cdot \underline{V} + \text{lower order terms.}$$

$$
(3) \qquad \begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - \underline{w} = 0, \\ \partial_t \underline{V} + (\underline{V} \cdot \nabla) \underline{V} + \mathfrak{a} \nabla \zeta = 0, \end{cases}
$$

From the previous analysis, the principal symbol of (3) should look like

$$
\begin{pmatrix} \underline{V} \cdot i\xi & i\frac{\xi^T}{|\xi|} \\ \mathfrak{a} i\xi & \underline{V} \cdot i\xi \end{pmatrix}.
$$

⇝ Imaginary eigenvalue of multiplicity two, with a Jordan block

The system is non strictly hyperbolic

⇝ Well-posedness requires therefore a Levy condition on the subprincipal symbol

$$
\text{(Rayleigh-Taylor)} \qquad \mathfrak{a} = -\frac{1}{\rho} \partial_z P_{|_{z=\zeta}} \geq a_0 > 0
$$

Theorem

Let $(\zeta^0, \nabla\psi^0)$ *be regular enough and*

$$\boxed{-H + \zeta^0 \geq h_{min} > 0} \quad \text{and} \quad \boxed{\mathfrak{a} = -\frac{1}{\rho}\partial_z P_{|_{z=\zeta}} \geq a_0 > 0.}$$

Then there exists a unique solution to (1) with initial condition (ζ^0, ψ^0) *on a time interval* $[0, T]$ *for some* $T > 0$.

Remark:

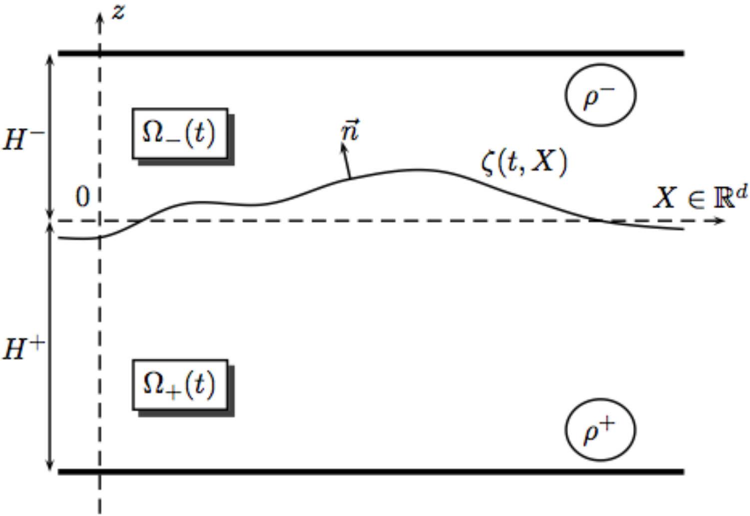
- The Rayleigh-Taylor may be automatically satisfied (infinite depth, flat bottom, . . .)
- Theorem true if $\sigma > 0$ with existence time $T_\sigma \to T$ as $\sigma \to 0$.

References (Local existence): Wu, Lindblad, L., Ambrose-Masmoudi, Coutand-Shkoller, Shatah-Zeng, Iguchi, Cordoba^2-Gancedo, Ming-Zhang, Rousset-Tzvetkov, Alazard-Burq-Zuily, etc.

I- General Introduction

2) The two fluids case

a) Notations



b) Free interface Euler equations

Equations in Fluid +

- $\mathrm{div}_{X,z} U^+ = 0$
- $\mathrm{curl}_{X,z} U^+ = 0$
- $\rho^+\big(\partial_t U^+ + (U^+ \cdot \nabla_{X,z})U^+\big) = -\nabla_{X,z} P^+ + \rho^+ \mathbf{g}.$

Equations in Fluid -

- $\mathrm{div}_{X,z} U^- = 0$
- $\mathrm{curl}_{X,z} U^- = 0$
- $\rho^-\big(\partial_t U^- + (U^- \cdot \nabla_{X,z})U^-\big) = -\nabla_{X,z} P^- + \rho^- \mathbf{g}.$

Equations at the interface

- $\boxed{[\![P^\pm]\!] = \sigma\kappa(\zeta)}$ with the notation $[\![A^\pm]\!] = A^+ - A-$.
- $\partial_t \zeta - \sqrt{1+|\nabla\zeta|^2} U_n^\pm = 0 \rightsquigarrow \boxed{U_n^+ = U_n^-}$

Equations at the rigid boundaries

- $U_n^+ = 0$ at $z = -H^+$
- $U_n^- = 0$ at $z = H^-$

c) Free interface Bernoulli equations

Equations in Fluid +

- $U^+ = \nabla_{X,z}\Phi^+$
- $\Delta_{X,z}\Phi^+ = 0$
- $\rho^+\left(\partial_t\Phi^+ + \frac{1}{2}|\nabla_{X,z}\Phi^+|^2\right) = -(P^+ - P_{atm}) - \rho^+ gz.$

Equations in Fluid -

- $U^- = \nabla_{X,z}\Phi^-$
- $\Delta_{X,z}\Phi^- = 0$
- $\rho^-\left(\partial_t\Phi^- + \frac{1}{2}|\nabla_{X,z}\Phi^-|^2\right) = -(P^- - P_{atm}) - \rho^- gz.$

Equations at the interface

- $[\![P^\pm]\!] = \sigma\kappa(\zeta)$
- $\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi^\pm = 0$

Equations at the rigid boundaries

- $\partial_n\Phi^+ = 0$ at $z = -H^+$
- $\partial_n\Phi^- = 0$ at $z = H^-$

Idea

Eliminate the pression from the Bernoulli equations

$$\begin{aligned} \rho^+\big(\partial_t\Phi^+ + \frac{1}{2}|\nabla_{X,z}\Phi^+|^2\big) &= -(P^+ - P_{atm}) - \rho^+ gz \\ \rho^-\big(\partial_t\Phi^- + \frac{1}{2}|\nabla_{X,z}\Phi^-|^2\big) &= -(P^- - P_{atm}) - \rho^- gz \end{aligned}$$

using the relation $\boxed{[\![P^\pm]\!] = \sigma\kappa(\zeta)}$.

$$\rightsquigarrow \psi = \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-$$

where $\boxed{\psi^\pm(t,X) = \Phi^\pm(t,X,\zeta(t,X))}$ and $\boxed{\underline{\rho}^\pm = \dfrac{\rho^\pm}{\rho^+ + \rho^-}}$.

Define Dirichlet-Neumann operators for the upper and lower fluids,

$$G^{\pm}\psi^{\pm} = G^{\pm}[\zeta]\psi^{\pm} := \sqrt{1+|\nabla\zeta|^2}\partial_n \Phi^{\pm}_{|_{z=\zeta}}$$

where

$$\begin{cases} \Delta_{X,z}\Phi^{\pm} = 0 \quad \text{in} \quad \Omega^{\pm}, \\ \Phi^{\pm}_{|_{z=\zeta}} = \psi^{\pm}, \qquad \partial_n \Phi^{\pm}_{|_{z=\mp H^{\pm}}} = 0. \end{cases}$$

CAUTION: Always UPWARD normal derivative!

With $g' = g(\underline{\rho}^{+} - \underline{\rho}^{-})$ the equations of motion then become

$$(4) \quad \begin{cases} \partial_t \zeta - G^{\pm}\psi^{\pm} = 0, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[\![\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]\!] - \frac{1}{2}\frac{[\![\underline{\rho}^{\pm}(G^{\pm}\psi^{\pm} + \nabla\zeta\cdot\nabla\psi^{\pm})]\!]}{1+|\nabla\zeta|^2} = -\frac{\sigma}{\rho^{+}+\rho^{-}}\kappa(\zeta) \end{cases}$$

$$(4)\quad \begin{cases} \partial_t \zeta - G^{\pm}\psi^{\pm} = 0, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[\![\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]\!] - \frac{1}{2}\frac{[\![\underline{\rho}^{\pm}(G^{\pm}\psi^{\pm}+\nabla\zeta\cdot\nabla\psi^{\pm})]\!]}{1+|\nabla\zeta|^2} = -\frac{\sigma}{\rho^{+}+\rho^{-}}\kappa(\zeta) \end{cases}$$

Comments:

- We recall that $\boxed{G^{+}\psi^{+} = G^{-}\psi^{-}}$.
- When $\zeta = 0$,

$$\boxed{G^{+}[0] = |D|\tanh(H^{+}|D|)} \quad \text{and} \quad \boxed{G^{-}[0] = -|D|\tanh(H^{-}|D|)}$$

- We need to express $\psi^{\pm}$ in terms of ζ and $\psi = \underline{\rho}^{+}\psi^{+} - \underline{\rho}^{-}\psi^{-}$.

e) Basic comments on the equations

The linearization of (4) around the rest state ($\zeta = 0, \psi^{\pm} = 0$) gives

$$\begin{cases} \partial_t \zeta - G^{\pm}[0]\psi^{\pm} = 0, \\ \partial_t \psi + g'\zeta = 0. \end{cases}$$

In order to express $\psi^{\pm}$ in terms of ζ and ψ, we solve

$$\begin{cases} |D|\mathrm{th}^{+}\psi^{+} + |D|\mathrm{th}^{-}\psi^{-} & = & 0 \\ \underline{\rho}^{+}\psi^{+} - \underline{\rho}^{-}\psi^{-} & = & \psi, \end{cases}$$

where $\boxed{\mathrm{th}^{\pm} = \tanh(H^{\pm}|D|)}$.

$$G^{+}[0]\psi^{+} = G^{-}[0]\psi^{-} \boxed{= G[0]\psi} = |D|\frac{\mathrm{th}^{+}\mathrm{th}^{-}}{\underline{\rho}^{+}\mathrm{th}^{-} + \underline{\rho}^{-}\mathrm{th}^{+}}\psi.$$

For the one fluid case, the linearized equation around $(0,0)$ was

$$(2) \qquad \partial_t^2\zeta + g|D|\tanh(H^+|D|)\zeta = 0.$$

The two fluids generalization is

$$(5) \qquad \partial_t^2\zeta + g'|D|\frac{\mathrm{th}^+\mathrm{th}^-}{\underline{\rho}^+\mathrm{th}^- + \underline{\rho}^-\mathrm{th}^+}\zeta = 0.$$

The deep water limit $H = \infty$

$$\partial_t^2\zeta + g'|D|\zeta = 0.$$

The shallow water limit $H^\pm \ll 1$

$$\partial_t^2\zeta - g'H\Delta = 0,$$

where $\boxed{H = \frac{H^+H^-}{\underline{\rho}^+H^- + \underline{\rho}^-H^+}}$.

f) One more formulation

As for the water waves, write the equations in ζ and $\underline{U}^{\pm} = U^{\pm}_{|_{z=\zeta}}$,

$$(6) \quad \begin{cases} \partial_t \zeta + \underline{V}^{\pm} \cdot \nabla \zeta - \underline{w}^{\pm} = 0, \\ \partial_t [\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!] + [\![\underline{\rho}^{\pm} (\underline{V}^{\pm} \cdot \nabla) \underline{V}^{\pm}]\!] + \mathfrak{a} \nabla \zeta = -\frac{\sigma}{\rho^+ + \rho^-} \nabla \kappa(\zeta), \end{cases}$$

with

$$\mathfrak{a} = -\frac{1}{\rho^+ + \rho^-} [\![\partial_z P^{\pm}_{|_{z=\zeta}}]\!] = g' + [\![\underline{\rho}^{\pm} (\partial_t + \underline{V}^{\pm} \cdot \nabla) \underline{w}^{\pm}]\!].$$

IMPORTANT: Write $[\![\underline{\rho}^{\pm} (\underline{V}^{\pm} \cdot \nabla) \underline{V}^{\pm}]\!]$ in terms of ζ and $[\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!]$:

$$[\![\underline{\rho}^{\pm} (\underline{V}^{\pm} \cdot \nabla) \underline{V}^{\pm}]\!] = \underbrace{[\![\underline{V}^{\pm}]\!] \cdot \nabla \langle \underline{\rho}^{\pm} \underline{V}^{\pm} \rangle}_{\rightsquigarrow \text{Kelvin-Helmholtz instability}} + \langle \underline{V}^{\pm} \rangle \cdot \nabla [\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!]$$

$\rightsquigarrow$ We need to find $\underline{V}^{\pm}$, $\underline{w}^{\pm}$ in terms of ζ and $[\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!]$.

Example: Linearization around $\zeta = 0$, $\underline{U}^\pm = (c^\pm, 0)$

$$(6)_1 \quad \partial_t \zeta + \underbrace{\underline{V}^\pm \cdot \nabla\zeta - \underline{w}^\pm}_{=-\sqrt{1+|\nabla\zeta|^2}U_n^\pm \sim -U_n^\pm} = 0$$

- As in the water waves case, let us write $\underline{w}^\pm \sim \mp \frac{\nabla}{|D|} \cdot \underline{V}^\pm$
- Therefore, $U_n^+ = U_n^- \sim -c^\pm \cdot \nabla\zeta \mp \frac{\nabla}{|D|} \cdot \underline{V}^\pm$
- We are therefore led to solve

$$\begin{cases} \frac{\nabla}{|D|} \cdot \underline{V}^+ + \frac{\nabla}{|D|} \cdot \underline{V}^- &= -[\![c^\pm]\!] \cdot \nabla\zeta \\ \underline{\rho}^+ \underline{V}^+ - \underline{\rho}^- \underline{V}^- &= [\![\underline{\rho}^\pm \underline{V}^\pm]\!]. \end{cases}$$

- We find

$$U_n^\pm \sim -(\underline{\rho}^+ c^+ + \underline{\rho}^- c^-) \cdot \nabla\zeta - \frac{\nabla}{|D|} \cdot [\![\underline{\rho}^\pm \underline{V}^\pm]\!]$$

$$(6)_2 \quad \partial_t [\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!] + [\![\underline{\rho}^{\pm} (\underline{V}^{\pm} \cdot \nabla) \underline{V}^{\pm}]\!] + \alpha \nabla \zeta = - \frac{\sigma}{\rho^{+} + \rho^{-}} \nabla \kappa(\zeta)$$

- We linearize
  $[\![\underline{\rho}^{\pm} (\underline{V}^{\pm} \cdot \nabla) \underline{V}^{\pm}]\!] \sim [\![c^{\pm}]\!] \cdot \nabla \langle \underline{\rho}^{\pm} \underline{V}^{\pm} \rangle + \langle c^{\pm} \rangle \cdot \nabla [\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!]$
- Proceeding as above, we find

$$[\![c^{\pm}]\!] \cdot \nabla \langle \underline{\rho}^{\pm} \underline{V}^{\pm} \rangle \sim \tfrac{1}{2} (\underline{\rho}^{+} - \underline{\rho}^{-}) [\![c^{\pm}]\!] \cdot \nabla [\![\underline{\rho}^{\pm} \underline{V}^{\pm}]\!] + \underbrace{\underline{\rho}^{+} \underline{\rho}^{-} \frac{1}{2} ([\![c^{\pm}]\!] \cdot \nabla)^2 \frac{\nabla}{|D|} \zeta}_{☠☠☠}$$

- We linearize $\boxed{\nabla \kappa(\zeta) \sim -\Delta \nabla \zeta}$

The linearized equations are therefore

$$\begin{cases} \partial_t \zeta + \widetilde{c} \cdot \nabla \zeta + \frac{\nabla}{|D|} \cdot [\![\underline{\rho}^\pm \underline{V}^\pm]\!] = 0 \\ \partial_t [\![\underline{\rho}^\pm \underline{V}^\pm]\!] + \widetilde{c} \cdot \nabla [\![\underline{\rho}^\pm \underline{V}^\pm]\!] + (\mathfrak{a} - \mathbf{KH} - \frac{\sigma}{\rho^+ + \rho^-} \Delta) \nabla \zeta = 0, \end{cases}$$

with

$$\boxed{\widetilde{c} = (\underline{\rho}^+ c^+ + \underline{\rho}^- c^-)} \quad \text{and} \quad \boxed{\mathbf{KH} = -\frac{1}{2} \underline{\rho}^+ \underline{\rho}^- \frac{([\![c^\pm]\!] \cdot \nabla)^2}{|D|} \geq 0}$$

⇝ Hyperbolicity condition:

$$\boxed{\mathfrak{a} + \tfrac{1}{2} \underline{\rho}^+ \underline{\rho}^- \tfrac{([\![c^\pm]\!] \cdot \xi)^2}{|\xi|} + \tfrac{\sigma}{\rho^+ + \rho^-} |\xi|^2 \geq 0}$$

⇝ Kelvin stability criterion

$$\boxed{g(\rho^+ - \rho^-) > \frac{(\rho^+ \rho^-)}{4\sigma(\rho^+ + \rho^-)^2} |[\![c^\pm]\!]|^4}$$

g) Nonlinear analysis

Theorem ($\sigma = 0$)

- *($d = 1$) The two fluids Euler equations are ill posed in any functional space B such that*

$$C^\infty \subsetneq B \subset C^{1+\alpha}.$$

- *They are well posed for analytic data.*

Proof.

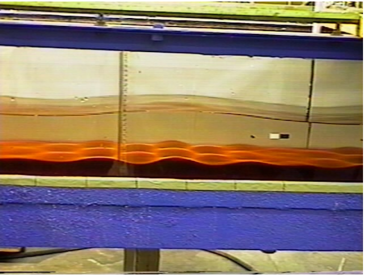
1) Iguchi-Tanaka-Tani, Lebeau, Kamotski-Lebeau, Wu, etc.
Because of the ellipticity of the symbol.
Indeed the previous analysis shows that (with $\widetilde{c} = 0$),

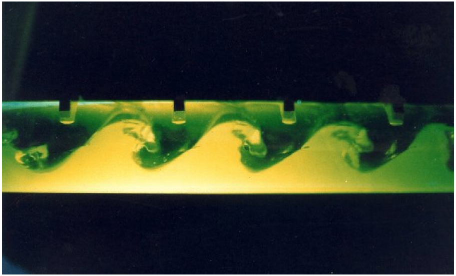
$$\partial_t^2 \zeta + \mathfrak{a}|D|\zeta + \frac{1}{2}\underline{\rho}^+\underline{\rho}^-|[\![c^\pm]\!]|^2\Delta\zeta = 0.$$

2) Duchon-Robert,Sulem^2, Sulem^2-Bardos: Cauchy-Kowaleski □

Problem:

Interfacial wave do exist!






Palawan
Tubbataha Reefs
Internal Waves

Theorem ($\sigma \neq 0$)

With surface tension, the equations are well-posed with Sobolev regularity over a time interval $[0, T_\sigma]$.

References: Iguchi-Tanaka-Tani, Ambrose-Masmoudi, Shatah-Zeng, Cheng-Coutand-Shkoller, Pusateri, etc.

Problem:

One has $T_\sigma \to 0$ as $\sigma \to 0$.

Question:

How to we recover the solution of the water waves equations without surface tension as $\sigma \to 0$ and $\underline{\rho}^- \to 0$?

Theorem (Pusateri)

Convergence to the solution of the water waves equations as $\sigma \to 0$ *and* $\underline{\rho}^- \to 0$ *provided that*

$$(\underline{\rho}^-)^2 \leq \sigma^{7/3}.$$

Problem:

This does not explain the observation of interfacial waves when $\underline{\rho}^- = \underline{\rho}^+ - \varepsilon$, $\varepsilon \ll 1$ (internal waves).

Question:

Such waves exist and are very well described by asymptotic models (shallow water, KdV, etc.). How can we explain their existence?

Remark: There is no surface tension in these asymptotic models.

Paradox:

Surface tension is necessary for the wave to exist but does not play any role on its dynamics.

Goal:

Find a criterion that

- Generalizes the Taylor criterion for two fluids interfaces
- Acts as a nonlinear version of Kelvin's criterion

Remark: The linear version of the Rayleigh-Taylor criterion $-\partial_z P > 0$ is $\boxed{g > 0}$.

$$(*) \qquad [\![-\partial_z P^{\pm}_{|z=\zeta}]\!] > \frac{1}{4} \frac{(\rho^+\rho^-)^2}{\sigma(\rho^+ + \rho^-)^2} c(\zeta^0) |[\![\underline{V}^{\pm}]\!]|^4$$

Theorem

Let $(\zeta^0, \nabla\psi^0)$ be regular enough and such that

$$H^{\pm} \mp \zeta^0 \geq h_{min} > 0 \qquad \text{and } (*) \text{ is satisfied}$$

Then there is a local solution to (4) on a time interval that depends on σ through (*) only.

Remark: Local solutions were known to exist; the theorem furnishes stable (in the sense of observability) solutions.

$$(*) \qquad [\![-\partial_z P^{\pm}_{|z=\zeta}]\!] > \frac{1}{4}\frac{(\rho^+\rho^-)^2}{\sigma(\rho^+ + \rho^-)^2} c(\zeta^0) |[\![\underline{V}^{\pm}]\!]|^4$$

$$(\mathrm{RT}) \qquad -\partial_z P_{|z=\zeta} > 0$$

$$(\mathrm{K}) \qquad g(\rho^+ - \rho^-) > \frac{(\rho^+\rho^-)}{4\sigma(\rho^+ + \rho^-)^2} |[\![c^{\pm}]\!]|^4$$

Corollary

Convergence to the solution of the water waves equations as $\sigma \to 0$ *and* $\underline{\rho}^- \to 0$ *provided that*

$$(\underline{\rho}^-)^2 \ll \sigma.$$

Question:

What about internal waves? Only hope:

$$[\![-\partial_z P^{\pm}_{|z=\zeta}]\!] \gg 1 \quad \text{or} \quad |[\![\underline{V}^{\pm}]\!]| \ll 1$$

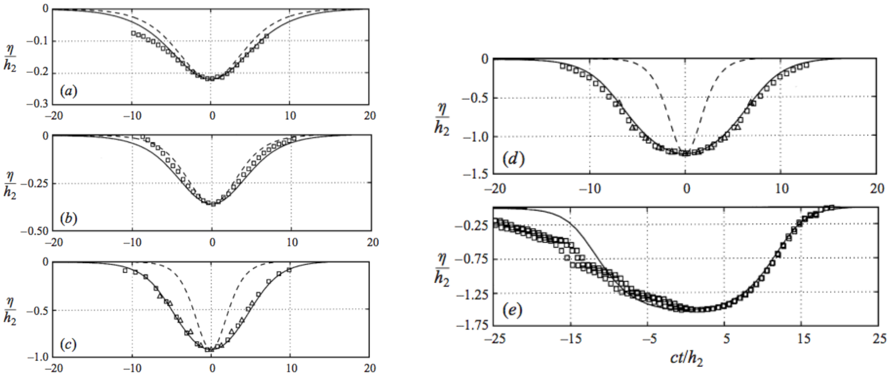
Problem:

How do we have an *a priori* knowledge of these quantities?

Goal:

We would like to be able to answer the question:

Is a wave of amplitude a and wavelength λ stable (observable)?



Grue et al. J. Fluid Mechanics, 1999.

It is possible to answer this question by asymptotic considerations. Let

- a: typical amplitude of the wave
- λ: typical wavelength
- $H = \frac{H^+H^-}{\underline{\rho}^+H^- + \underline{\rho}^-H^+}$: typical depth

and

- $\mu = \frac{H^2}{\lambda^2}$: shallowness parameter
- $\Upsilon = (\underline{\rho}^+\underline{\rho}^-)^2 \frac{a^4}{H^2} \frac{(\rho^+ + \rho^-)g'}{4\sigma}$: "stability" parameter

Theorem

The result of the previous theorem is uniformly true as $\mu \to 0$ (shallow water regime).
A practical *stability criterion is then*

$$\Upsilon \ll 1$$

II- The Stability Theorem

1) Rigorous derivation of the equations

In order to study

$$(1) \quad \begin{cases} \partial_t \zeta - G^{\pm}\psi^{\pm} = 0, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[\![\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]\!] - \frac{1}{2}\frac{[\![\underline{\rho}^{\pm}(G^{\pm}\psi^{\pm}+\nabla\zeta\cdot\nabla\psi^{\pm})]\!]}{1+|\nabla\zeta|^2} = -\frac{\sigma}{\rho^+ + \rho^-}\kappa(\zeta), \end{cases}$$

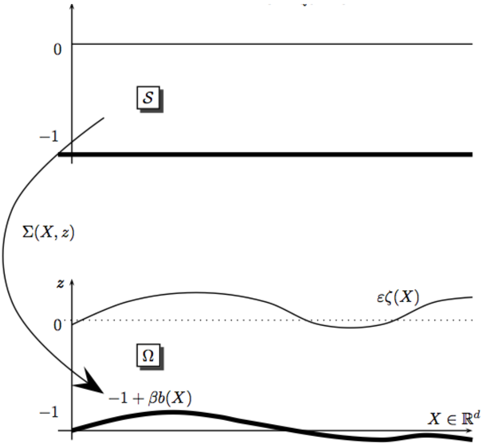
we need to express $\psi^{\pm}$ in terms of ζ and $\psi = \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-$.

<u>Remark</u>: One easily checks that

$$G^{\pm}[\zeta, H^{\pm}] = \frac{1}{H^{\pm}} G^{\pm}[\frac{1}{H^{\pm}}\zeta, 1].$$

$\rightsquigarrow$ In the analysis of the DN operators, we take $H^{\pm} = 1$.

a) Some facts on the DN operator



$$H^{\pm} \pm \zeta \geq h_{min}$$

We write $\phi = \Phi \circ \Sigma$; then

$$\Delta_{X,z}\Phi = 0 \text{ in } \Omega \quad \rightsquigarrow \quad \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z}\phi = 0 \text{ in } \mathcal{S}.$$

We have the following properties:

- (P1) $\boxed{G^+\psi = \partial_n \phi_{|_{z=0}}}$ $\big(= \mathbf{e}_z \cdot P(\Sigma)\nabla_{X,z}\phi_{|_{z=0}}\big)$
- (P2) G^+ is positive and symmetric for the L^2-scalar product
- (P3) $\zeta \mapsto G^+[\zeta]\psi$ is analytic and

$$\boxed{dG^+(h)\psi = -G^+(h\underline{w}^+) - \nabla \cdot (h\underline{V}^+)}$$

Remark: Similar properties also hold for G^-

Definition (Beppo-Levy spaces)

For all $s \geq 0$, we define

$$\dot{H}^{s+1} = \{f \in L^2_{loc}(\mathbb{R}^d), \nabla f \in H^s(\mathbb{R}^d)^d\}/\mathbb{R}.$$

Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$. Then, for all $0 \leq s \leq t_0 + 1$,

- (P4) $G^+[\zeta] : \dot{H}^{s+1/2} \to H^{s-1/2}$ and

$$\boxed{|G^+\psi|_{H^{s-1/2}} \leq M|\nabla\psi|_{H^{s-1/2}}} \quad = M|\psi|_{\dot{H}^{s+1/2}}.$$

- (P5) For all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}$,

$$\boxed{|(\Lambda^s G^+\psi_1, \Lambda^s\psi_2)| \leq M|\nabla\psi_1|_{H^{s-1/2}}|\nabla\psi_2|_{H^{s-1/2}}}$$

- (P6) If $\psi^\pm = \phi^\pm_{|_{z=0}}$ then

$$\boxed{|\psi^\pm|_{\dot{H}^{s+1/2}} \leq M\|\Lambda^s\nabla_{X,z}\phi^\pm\|_2}$$

<u>Notation</u>: $M = M(\frac{1}{h_{min}}, |\zeta|_{H^{t_0+2}})$.