NOTES ON THE COURSE DIFFERENTIAL INCLUSIONS AND THEIR GEOMETRY.

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Abstract. We recollect here the notes of the course Differential inclusions and their geometry given by Bernd Kirchheim at the ICMAT, Madrid.

Contents

1. **Introduction.**

These are the notes for a course of four lectures, given by Bernd Kirchheim in September 2010 at the ICMAT, Madrid. This course was included in the trimester *Calculus of Variations, Singular Integrals and Incompressible flows*.

As appears in the index, in Section 2 we start by introducing the basic tools in order to develop the theory, the concept of rank-one connections and rigidity of a set, which will lead us to the construction of the Tartar square, as a counterexample for the rigidity of approximate solutions. In Section 3, the concepts of rank-one convex, quasiconvex, polyconvex and convex functions are given and we finish the section by stating a question about rigidity of a finite set $\mathcal{K}$ without rank-one connections, which will be the object of the study of the Sections 4 and 5. In Section 6, we study the relation between some kind of solutions of the Euler-Lagrange system and the solutions of a partial differential inclusion (PDI) derived from the partial

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differential equations. Finally we finish the schedule by considering the relation between the notion of rank-one convexity and the L^1 non inequality due to Ornstein.

I want to mention that all the material is extracted from the following references, included at the end of the review. For Sections 2, 3, 4 and 5, see [Kir] and [KS]. Concerning Section 6, the basic reference is [Sz] and for Section 7, the basic papers that might be considered are [BB], [Mc] and [Or]. I apologize in advance for the obviously incomplete bibliography, but being sure that the reader will be able to “fill in the blanks” from the cited references.

Acknowledgments. I wish to thank the organizers for inviting me and for giving me the oportunity to make this notes, especially to Daniel Faraco. I also want to express mi gratitude to Angel Castro for his dedication and his patience. And finally but not less important, to Bernd Kirchheim.

2. **Rank-one connections and rigidity.**

We denote by $[Du]$ the essential range of the gradient of u, i.e. the smallest closed subset of $\mathbb{R}^{n\times m}$ such that $Du(x) \in [Du]$ for almost every $x \in \Omega$.

This issue plays a role in the study of material microstructure, and is linked to the question of existence and regularity of solutions to partial differential inclusions of the type

$$Du(x) \in K \quad \text{a.e.} \quad x \in \Omega,$$

where $K \subset \mathbb{R}^{m\times n}$ is a prescribed (compact) set of matrices.

The following construction is well known: let $A, B \in \mathbb{R}^{m\times n}$ be two matrices such that $\operatorname{rank}(A-B) = 1$, so that

$$A - B = a \otimes n \tag{2.1}$$

for some vectors $a \in \mathbb{R}^m$ and $n \in \mathbb{R}^n$.

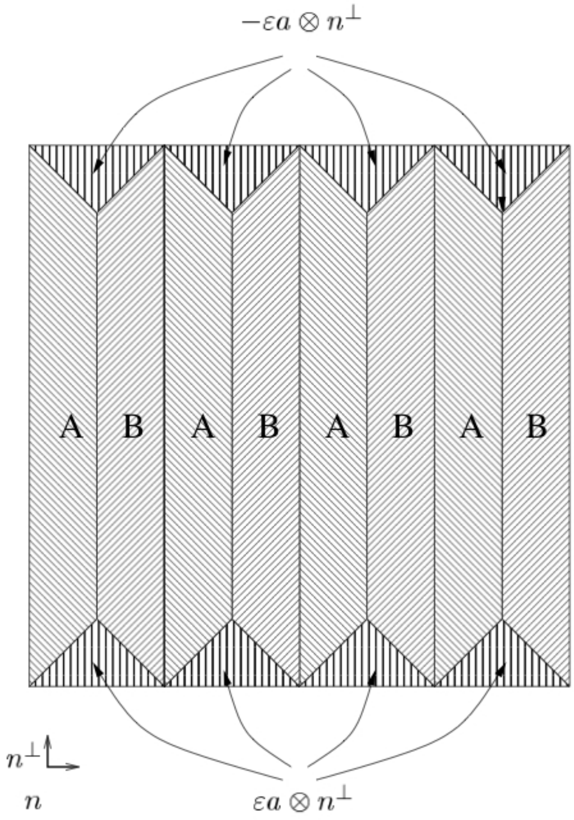
For any Lipschitz profile $h : \mathbb{R} \to \mathbb{R}$ with $h'(t) \in \{0, 1\}$ a.e., the map

$$u(x) = Bx + ah(x \cdot n), \tag{2.2}$$

is a Lipschitz map whose gradient takes the values A or B almost everywhere. This type of example is called a *simple laminate*, and whenever two matrices A, B satisfy $\operatorname{rank}(A - B) = 1$, one speaks of a *rank-one connection*.

It is also well known that if $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A - B) > 1$, then the only Lipschitz maps with gradient $Du(x) \in \{A, B\}$ a.e. are affine maps. Moreover, in [BJ] J. M. Ball and R. D. James established the much stronger statement that whenever $\{u_j\}$ is a sequence of maps bounded in $W^{1,1}$ such that dist $(Du_j, \{A, B\}) \to 0$ in L^1 strongly, then - up to a subsequence - $Du_j \to A$ or $Du_j \to B$ strongly in L^1.

The meaning of condition (2.1) is that the linear mappings corresponding to A and B agree along a whole hyperplane of $\mathbb{R}^m$. Therefore, domains on which an affine map has gradient A can touch with domains on which the gradient equals B along this hyperplane without violating the tangential continuity. Then it is of course possible to construct a Lipschitz map on a connected domain that uses precisely the gradients A and B. Each linear map corresponding to a matrix C in the rank-one segment $[A, B] \subset \mathbb{M}^{n\times m}$ can be, up to a quickly oscillating but arbitrary small error, obtained as the macroscopic behaviour of a map with microstructure $\{A, B\}$. Using an arbitrary small interpolation layer but making the construction scale sufficiently fine, we can realize the boundary datum C precisely and still keep the Lipschitz constants uniformly bounded. Viewing this from the outside, it means that we can without changing the boundary datum modify the gradient distribution which originally was a Dirac sitting in C into distribution which up to an arbitrarily small error lives at the end points of a freely chosen rank-one segment passing through C. The preservation of the boundary data implies, due to Fubini's theorem, that during this modification the barycentre of the distribution does not change.



The picture on the left illustrates the value of the gradient in the different parts of the domain. After adding a suitable affine function we can suppose $C = 0$ for the approximated matrix. The size of the error in the interpolation layer as well as the area of the latter becomes arbitrarily small if the finess of the construction increases. The gradient distributions before and after the distribution look as pictured below.

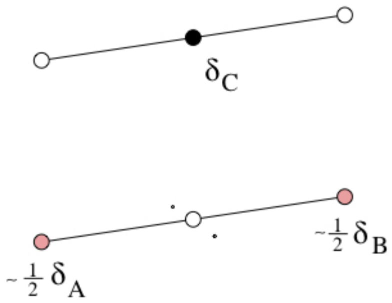


Figure 1. Lamination and gradient distributions

Of course, the question arises whether this simple, so called lamination construction is very special or if it already covers or at least in some sense generates the whole spectrum os possible constructions. To ask more precisely, we introduce the following notions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. The gradient distribution of a Lipschitz map $f : \Omega \to \mathbb{R}^n$ is the probability measure on the space of all $(n \times m)$-matrices given by

$$\mathcal{M} \subset \mathbb{M}^{n\times m} \to \mathscr{H}^m(\{x \in \Omega : \nabla f(x) \in \mathcal{M}\})/\mathscr{H}^m(\Omega) \text{ for all } \mathcal{M} \subset \mathbb{M}^{n\times m}.$$

A compactly supported probability μ on $\mathbb{M}^{n\times m}$ is an approximate gradient distribution if μ is the weak* limit of the gradient distributions of a sequence $\{f_k\}$ of uniformly Lipschitz mappings that are defined on a fixed domain and all have the same affine boundary data.

We say that the compact set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ is

- *rigid for exact solutions* if each on a domain defined Lipschitz map f with $\nabla f \in \mathcal{K}$ a.e. is necessarily affine. Equivalently, each exact gradient distribution with support in $\mathcal{K}$ is a Dirac measure.
- *rigid for approximate solutions* if each approximate gradient distribution living in $\mathcal{K}$ is a Dirac measure.

The just defined approximate gradient distributions agree with the in the literature more frequently studied Gradient Young measure $\mathscr{M}_{qc}$. We also consider their localization $\mathscr{M}_{qc}(\mathcal{K}) = \{\mu \in \mathscr{M}_{qc}\,;\, \mathrm{spt}(\mu) \in \mathcal{K}\}$ and emphasize that for a given $\mu \in \mathscr{M}_{qc}(\mathcal{K})$ the approximating exact distributions do not need to have supports in $\mathcal{K}$ but only close to it. This is precisely what characterizes them as minimizing sequence of a corresponding variational problem. Another important aspect is, that mappings whose gradient distributions generate measure in $\mathscr{M}_{qc}$ respect the same affine boundary datum. This enables us to extend such maps in a periodic way and, after suitable rescalings, to obtain highly oscillating and hence weakly converging gradients and puts approximate gradient distributions in perfect duality with suitable convexity notions on the space of matrices.

In fact, our remark about the barycentre of the gradient distribution already implies that all possible macroscopical boundary values are contained in the convex hull os the microscopical gradients. In the scalar case this statement cannot be strengthened. In the situation of higher dimensions, however, nonlinear Null-Lagrangians do exists. There are mappings $F : \mathbb{M}^{n\times m} \to \mathbb{R}$ such that $\int_\Omega F(\nabla u(x))\,dx$ is determined by $u|_{\partial\Omega}$ already. We denote this class by $\mathcal{NL}(n,m)$ an it consists precisely of all affine combinations of the minors of $(n\times m)$-matrices. Consequently, each approximate gradient distribution also satisfies

$$F(\bar{\mu}) = \int F(X)\,d\mu(X)\,\text{for all}\,F \in \mathcal{NL}(n,m).$$

We write $\mathscr{M}_{pc}$ for the family of all measures which satisfy these so-called minor conditions because they also satisy Jensen's inequality not only for convex, but also for all polyconvex functions. Such functions are the pointwise supremum over some family of Null-Lagrangians or, equivalently, can be written as a convex function of all the minors. Therefore, $\mathscr{M}_{pc}$ gives an upper bound for $\mathscr{M}_{qc}$ and it is clear that microstructures in $\mathcal{K}$ have their boundary values in $\mathcal{K}^{pc} = \{\bar{\mu}; \mu \in \mathscr{M}_{pc}(\mathcal{K})\}$. More interesting, a Hahn-Banach argument gives the dual representation

$$\mathcal{K}^{pc} = \{X; F(X) \leq \sup F(\mathcal{K})\ \text{for all}\, F : \mathbb{M}^{n\times m} \to \mathbb{R}\,\text{polyconvex}\}$$

The originally derived duality concerned approximate gradients distributions and quasiconvex functions $F : \mathbb{M}^{n\times m} \to \mathbb{R}$, which are characterized by fulfilling

$$\int_\Omega F(A + \nabla\varphi(y))\, dy \geq \int_\Omega F(A)\, dy \,\text{for all}\, A \in \mathbb{M}^{n\times m} \,\text{and}\, \varphi \in \mathrm{Lip}_0(\Omega, \mathbb{R}^n).$$

It is quite easy to see that the condition is independent of the domain considered. D. Kinderlehrer and P. Pedregal showed that approximate gradient distributions are among probabilities distinguished by satisfying Jensen's inequality for all quasiconvex functions. Expressed formally, we have

$$\mathscr{M}_{qc} = \{\mu; \mathrm{spt}(\mu)\,\text{compact and}\, \int F(Y)\, d\mu(Y) \geq F(\bar{\mu})$$
$$\text{for all}\, F : \mathbb{M}^{n\times m} \to \mathbb{R}\,\text{quasiconvex}\},$$

and again the quasiconvex hull $\mathcal{K}^{qc}$ which now consists only of the really possible boundary values among all of those given by $\mathcal{K}^{pc}$ can be described in terms of separation properties of quasiconvex functions. Since the definition of quasiconvexity in mucho more difficult to verify, a good inner estimate for $\mathcal{K}^{qc}$ is needed.

Here the already introduced lamination construction comes into the game. It shows that $\mathcal{K}^{qc}$ at least contains $\mathcal{K}^{lc}$. The latter is the smallest superset of $\mathcal{K}$ which together with two rank-one connected matrices A and B contains also the rank-one segment $[A, B]$. But what happens in the case that there exists no rank-one connection at all in $\mathcal{K}$?. More precisely, we ask:

Let $\mathcal{K} \in \mathbb{M}^{n\times m}$ be compact and suppose $\mathrm{rank}(A - B) \neq 1$ for all $A, B \in \mathcal{K}$. Does this imply that $\mathcal{K}$ in necessarily rigid for approximate or exact solutions?

We want to stress that these two rigidity notions are independent of each other. The seemingly plausible implication that rigidity for approximate solution implies also rigidity for exact ones is refuted by the example of holomorphic functions whose gradient run trough an infinite compact subset K of the space of all conformal matrices $\{F \in \mathbb{M}^{2\times 2}; F_{11} = F_{22}, F_{12} = -F_{21}\}$. On the other hand, this space does not contain rank-one connections and is rigid for approximate solutions.

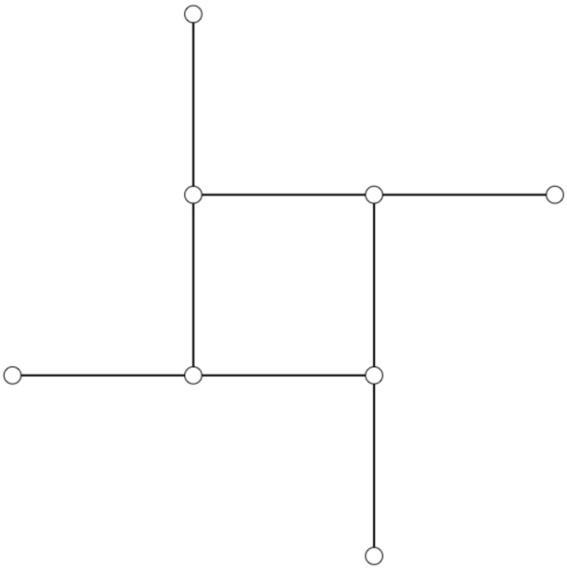
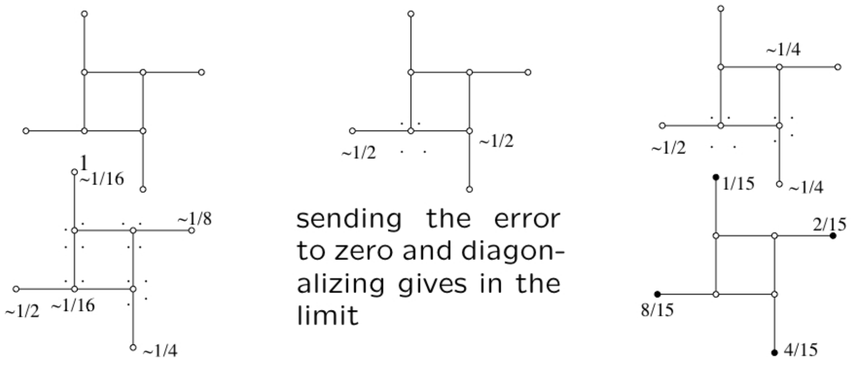


FIGURE 2. Tartar square.

The first, and moreover finite counterexample concerning rigidity of approximate solutions was given by L. Tartar in 1983, see [Tar]. The whole construction takes place in the diagonal (2×2)-matrices. In order to illustrate it, we identify the point $(x, y) \in \mathbb{R}^2$ with the $\mathrm{diag}(x, y) \in \mathbb{M}^{2\times 2}$. Therefore, rank-one connections precisely correspond to vertical or horizontal segments in the plane. Using this, it is easily checked that the set consisting of the four matrices

$$\begin{aligned} A_1 &= \mathrm{diag}(-3,-1), A_2 = \mathrm{diag}(1,-3), \\ A_3 &= \mathrm{diag}(3,1), \quad A_4 = \mathrm{diag}(-1,3), \end{aligned}$$

does not have any rank-one connection. If we now start from a Dirac measure in $\mathrm{diag}(-1,1)$, i.e. the affine map $(x_1, x_2) \to (-x_1, x_2)$, and perform the lamination construction, then we gradually obtain the exact gradient distributions shown on the picture. All of them have again the same barycentre, and in the weak* limit they give an approximate gradient distribution supported in $\mathcal{K}_T = \{A_1, \ldots, A_4\}$ which is, therefore, not a Dirac measure. This shows that the "Tartar square $\mathcal{K}_T$" is not rigid for approximate solutions.



This example also makes clear that the class of prelaminates (also called laminates of finite order) which is the class of probabilities that can be obtained from Dirac measures via the just shown iterative splitting procedure in rank-one directions plays, together with its weak* closure $\mathscr{M}_{rc}$, a very important role. Because this construction cares only about ran-one directions, it is natural to expect that $\mathcal{M}_{rc}$ is in duality with the cone of rank-one convex functions. These are the $F : \mathbb{M}^{n\times m} \to \mathbb{R}$ which are convex along the rank-one lines. In the paper [Ped] this and the two dual representations of the rank-one convex hull $\mathcal{K}^{rc}$ were established. Since all the quasiconvex functions are rank -one convex, it is clear that

$$\mathcal{K}^{rc} = \{X : F(X) \leq \sup F(\mathcal{K}) \text{ for all } F : \mathbb{M}^{n\times m} \to \mathbb{R} \text{ rank-one convex}\}$$

provides an inner and moreover often numerically manageable approximation for $\mathcal{K}^{qc}$.

Naive strategy to solve PDI:

Find $\{f_k\}$ such that

$$\int \operatorname{dist}(\nabla f_k, K) \to 0, \tag{2.3}$$

and hope the limit f be a solution of the PDI $\nabla f \in K$. This is not true in general. ∇f_k converges weakly, but not strongly.

Definition 2.2. We say that f is a piecewise affine function (pwa) if $\exists \Omega_i$ disjoint such that $\Omega \subset \cup\Omega_i$, $|\Omega \setminus \cup \Omega_i| = 0$ and $\forall i$ $f|_{\Omega_i}$ is affine.

Definition 2.3. A PDI $\nabla u \in \mathcal{K}$ is said to be locally improvable in $\mathcal{U}$ if $\forall A \in \mathcal{U}$, $\epsilon > 0$, there exists $u : \Omega \to \mathbb{R}^m$ Lipschitz such that:

(a) $u(x) = Ax$, $\forall x \in \partial\Omega$ ("preserve affine boundary data").
(b) $\int_\Omega \operatorname{dist}(\nabla u(x), \mathcal{K})\, dx < \epsilon|\Omega|$ ("nearly a solution").
(c) u is piecewise affine, $\nabla u \in \mathcal{U}$ a.e.

Sychev-Müller-Šverák: We say that "$\mathcal{U}$ can be reduced to $\mathcal{K}$", show that $\forall A \in \mathcal{U}$, Ω open, there are Lipschitz maps u with $u(x) = Ax$ on $\partial\Omega$ and $\nabla u \in \mathcal{K}$ a.e.
Dacorogna-Marcellini: We say that "$\mathcal{U}$ has relaxation property wrt $\mathcal{K}$", if $\mathcal{U}$ also quasiconvex then there are many exact solutions.

Definition 2.4. Gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall A \in \mathcal{U}$ with $\operatorname{dist}(A, \mathcal{K}) > \epsilon$, there exists a piecewise affine function v satisfying $\nabla(A + v) \in \mathcal{U}$, $v \equiv 0$ on $\partial\Omega$ and $\int_\Omega |\nabla u| > \delta|\Omega|$.

Theorem 2.1. *Let $\mathcal{U}$ be bounded and the gradients in $\mathcal{U}$ be stable only near $\mathcal{K}$. If $A \in \mathcal{U}$, then the generic functions in the L^∞-closure of*

$$X = \{u \,\text{piecewise affine} : \nabla u \in \mathcal{U}, u|_{\partial\Omega} = A\},$$

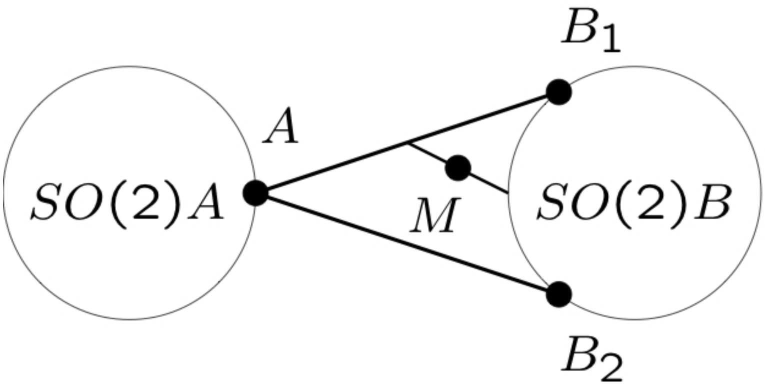
solve $\nabla u \in \mathcal{K}$.

Two well problem $\mathcal{K} = SO(2)A \cup SO(2)B$.

Let us consider the following matrices

$$A = \begin{pmatrix} a & \\ & b \end{pmatrix}, \qquad B = \begin{pmatrix} b & \\ & a \end{pmatrix}, \qquad 0 < a < b.$$

Then A, and by group invariance any $A' \in SO(2)A$ has two rank-one connections into $SO(2)B$.



Let us define

$$\mathcal{K}_{l+1} = \cup\{[X, Y] : X, Y \in \mathcal{K}_l \quad \text{rk-1 con.}\},$$

$\mathcal{K}_0 = \mathcal{K}$ are "lamination hulls" of $\mathcal{K}$, $\mathcal{K}^{lc} = \bigcup_l^\infty \mathcal{K}_l$.

3. **Convexity notions for vectorial variational problems.**

Definition 3.1. For a given function $f : \mathbb{M}^{n\times m} \to \mathbb{R}$, we say that

a) f is rank-one convex if for each $A \in \mathbb{M}^{n\times m}$ and all $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ the function $t \to f(A + t \cdot a \otimes b)$ is convex on the real line.

b) f is quasiconvex if for each open bounded subset Ω of $\mathbb{R}^m$, each $A \in \mathbb{M}^{n\times m}$ and any $\varphi \in \mathrm{Lip}_0(\Omega, \mathbb{R}^n)$ the Jensen inequality
$$f(A) \leq \frac{1}{\mathscr{L}^m(\Omega)} \int_\Omega f(A + \nabla u(x))dx$$
holds.

c) f is polyconvex if there is a convex function $g : \mathbb{R}^{\tau(n,m)} \to \mathbb{R}$ such that $f(X) = g(\hat{X})$ for all $X \in \mathbb{M}^{n\times m}$. Here $\hat{X}$ is the minor vector consisting of all sub-determinants (of order one up to $\min(n, m)$) of X and has the easily computable length $\tau(n, m)$.

d) f is convex if for all $A, B \in \mathbb{M}^{n\times m}$ the function $t \to f(A + t \cdot B)$ is convex on the real line.

Then, it is well-known that

f convex$\Rightarrow$ f polyconvex$\Rightarrow$ f quasiconvex$\Rightarrow$ f rank-one convex.

$\mathcal{K} \subset \mathbb{R}^{n\times m}$ is said to be rigid if given $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz such that $\nabla u \in \mathcal{K}$ a.e. implies that u is (locally) affine.

We have that $\mathcal{K}$ rigid $\Rightarrow$ no R1C. The converse is not true, as can be seen for conformal matrices (u holomorph).

This leads to a natural question proposed by J.M.Ball and R.James in 1990. Let $\mathcal{K}$ be finite without R1C. Is $\mathcal{K}$ necessarily rigid?

The answer to this question depends on the cardinal of the set $\mathcal{K}$.

- If card($\mathcal{K}$)=2 the answer is yes.

- If card($\mathcal{K}$)=3 the answer is yes, as was shown by Šverák and Zhang.

- If card($\mathcal{K}$)=4 the answer is yes and it was proved by Chlebik and Kirchheim. The crucial case was the following one.
  If $\mathcal{K} \subset \mathrm{Symm}(2 \times 2) \cap \{\det = -1\}$, then $\exists p : u = \nabla p$, $\det(D^2 p) = -1$.
  $\Rightarrow$ hyperbolic Monge-Ampère equation leads to good understanding of all maps with 4 gradients (exact solutions).

- If card($\mathcal{K}$)=5 the answer is no, as it was proved by Preiss and Kirchheim. This implies that it seems to be a general jump in the complexity of contructable maps.

4. **The four gradient problem.**

In this section we show that the four point configurations can never form a properly non-rigid set. We previously give some lemmas which will be used during the proof. All the lemmas and the theorem appear in [CK] and [Kir]. The approach we will use is the same. Firstly, after a dimension reduction argument given in lemma 4.1, in lemma 4.2 we state some facts from linear algebra which allow a further reduction of the cases necessary to be considered. In lemma 4.3 the degenerate case is treated.

Lemma 4.1. *Let* $\tilde{A} \subset \mathbb{M}^{n\times m}$ *be finite and without rank-one connections. Suppose there exists a nonaffine lipschitz map on a domain* $\tilde{f} : \tilde{\Omega} \subset \mathbb{R}^m \to \mathbb{R}^n$ *such that* $\nabla \tilde{f} \in \tilde{A}$ *almost everywhere. Then there are* $g \in \mathbb{M}^{2\times m}$ *and* $h \in \mathbb{M}^{n\times 2}$ *and a lipschitz map* f *from* $[0,1]^2$ *into* $\mathbb{R}^2$ *such that*

- $\mathcal{A} = \{h \circ A \circ g : A \in \tilde{A}\}$ *does not contain any rank-one connection.*
- $\nabla f \in \mathcal{A}$ *almost everywhere and* f *is nonaffine.*

Lemma 4.2. *Consider* 2×2 *matrices* $A_1 = 0$, $A_2 = Id$, A_3 *and* A_4 *with* $\min_j det(A_j) < 0$. *Then one of the following two cases occurs.*

- *There are* $v, w \in \mathbb{S}^1$ *such that* $w || A_j v$ *for* $j = 1, \dots, 4$.
- *The exists a regular matrix* $P \in \mathbb{M}^{2\times 2}$, *a matrix* $S \in \mathbb{M}^{2\times 2}_{sym}$ *and a real* D *such that* $PA_j - S$ *is symmetric and* $det\{PA_j - S\} = D$ *for* $j = 1, \dots, 4$.

Lemma 4.3. *Assume* $\mathcal{A} \subset \mathbb{M}^{2\times 2}$ *does not contain any rank one connection and that there are* $v, w \in \mathbb{S}^1$ *with* $w || Av$ *for all* $A \in \mathcal{A}$. *If the lipschitz map* $f : [0,1]^2 \to \mathbb{R}^2$ *satisfies* $\nabla f \in \mathcal{A}$ *almost everywhere, then* f *in necessarily affine.*

Theorem 4.4. *Let us be given four matrices* $A_1, \dots, A_4 \in \mathbb{M}^{n\times m}$ *with* $rank\{A_i - A_j\} \neq 1$ *for all* i, j. *If* $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^m$ *a domain, is a lipschitz map with* $\nabla f(x) \in \{A_1, \dots, A_4\}$ *almost everywhere, then* f *is necessarily affine.*

CONSTRUCTION. Even it has been shown that all non-rigid 4 point configurations are trivial, it can be identified which of the non-trivial 4 point configurations might be the “most non-rigid”.

Let us consider the hyperboloid of the type $\mathcal{H}_D = \{M \in \mathbb{M}^{2\times 2}_{sym} : \det(M) = D\}$. The geometry of the one-sheeted hyperboloid $\mathcal{H}_D$ allows the following improvement of the classical “Tartar square”. In fact, four points in $\mathcal{H}_D$ can in the same moment generate four different squares.

To be more precise, we identify via (restricted) conformal coordinates

$$M = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = z_{\mathcal{H}} + (x+iy)_{\mathcal{H}} \sim p = (x,y,z) \in \mathbb{R}^3,$$
$$\det(M) \sim \widehat{\det}(p) = z^2 - x^2 - y^2.$$

Now, it is easily checked that given $p = (x,y,z) \in \mathcal{H}_{-1}$ the two lines in direction $(a,b,1)$, where $(a+ib) = (z \pm i)/(x - iy)$, through p are rank-one lines staying inside the hyperboloid. We choose $P_1 = (x,y,z)$ and $P_2 = (-x,-y,z)$, $z > 1$ on $\mathcal{H}_{-1}$, then it in not difficult to verify that the line l_1^u through p_1 in direction $\left(\frac{z+i}{x-iy}, 1\right)$ and the line l_3^u through p_2 in direction $\left(\frac{-z+i}{x-iy}, 1\right)$ intersect in some point $R \in \mathcal{H}_1$ with $\langle R, e_3 \rangle < z$. Symmetry reasons ensure that also l_2^u through P_1 in direction $\left(\frac{z-i}{x-iy}, 1\right)$ and l_4^u through P_2 in direction $\left(\frac{-z-i}{x-iy}, 1\right)$ intersect in $R' \in \mathcal{H}_{-1}$. A simple calculation gives that $\langle R, e_3 \rangle = \langle R', e_3 \rangle = z - \left((x^2 + y^2)/z\right) = -1/z > -z$. We say that that the polygon P_1, R, P_2, R', P_1 forms the upper boundary and repeat the same construction in the lower part of the hyperboloid. However , in order to avoid the intersection of the lower and the upper boundary we rotate the lower part by 90° in the xy-plane. So $P_3 = (-y, x, -z)$ and $P_4 = (y, -x, -z)$.

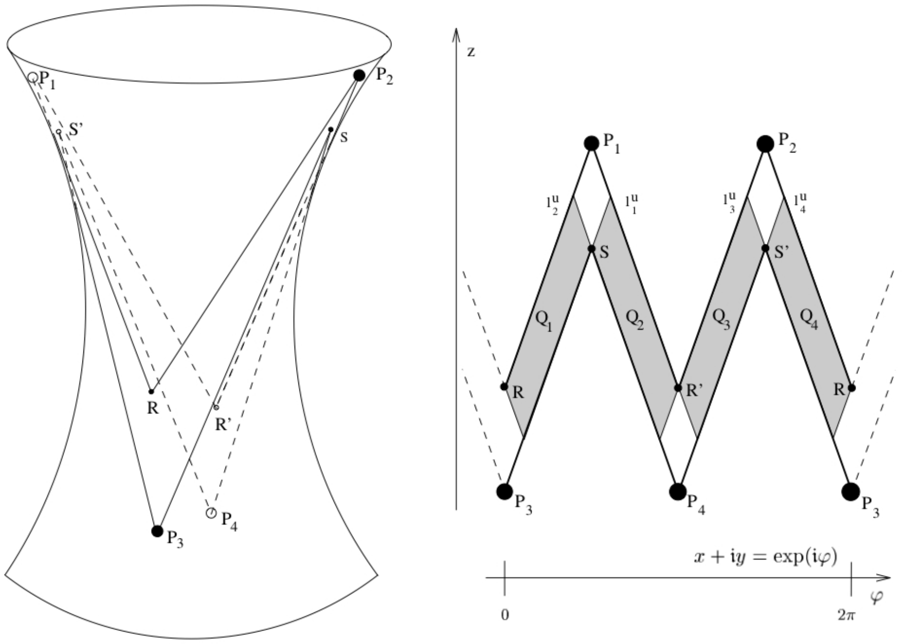
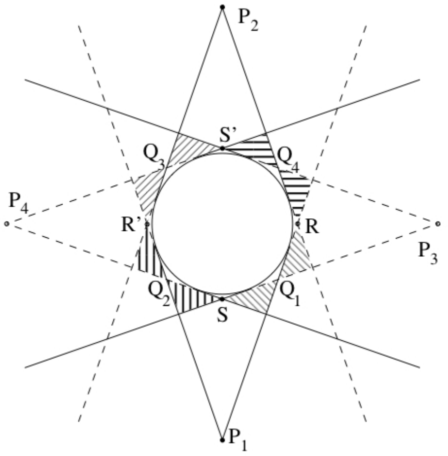


Figure 3. The fourfold Tartar configuration.

The picture above shows that choosen the points $P_1, \ldots, P_4$ on the hyperboloid give 4 Tartar squares.

We can also visualize the fourfold-Tartar configuration in a different way. We consider the orthogonal projection into the xy-plane, so the waist of the hyperboloid becomes the inner circle in the picture. Rank-one segments in the upper half of the hyperboloid $\mathcal{H}_1$ are depictured as full lines, those in the lower part dashed. The fourfold Tartar squares twist into four “butterflies”.



5. **The five gradient problem.**

Here we will present a non-rigid five points configuration in $\mathbb{M}^{2\times 2}_{sym}$ without rank-one connections which was discovered by Bernd Kirchheim and David Preiss (see [KP]). Moreover, it tuns out that these properties of the configuration are stable under small perturbations inside $\mathbb{M}^{2\times 2}_{sym}$. As in the previous section we use the identification of $\mathbb{M}^{2\times 2}_{sym}$ and $\mathbb{R}^3$ via (restricted) conformal coordinates. So we set

$$M = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = z_{\mathcal{H}} + (x+iy)_{\mathcal{H}} \sim p = (x,y,z) \in \mathbb{R}^3,$$

$$\det(M) \sim \widehat{\det}(p) = z^2 - x^2 - y^2.$$

We use the short and more intuitive notation $[A_1, \dots, A_k]$ for the convex hull of the set $\{A_1, \dots, A_k\}$.

CONSTRUCTION. Choose the following left, right, front and back points one the one-sheeted hyperboloid $\mathcal{HYP} = \mathcal{H}_1 = \{(x,y,z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = -1\}$:

$$P_{L_0} = (0, -2, \sqrt{3}),\ P_{R_0} = (0, 2, \sqrt{3}),\ P_{F_0} = (2, 0, -\sqrt{3}),$$

$$\text{and}\, P_{B_0} = (-2, 0, -\sqrt{3}).$$

Suppose now that P_L, P_R, P_F and P_B are points on $\mathcal{HIP}$ sufficiently close to their corresponding original points just selected. We are going to define various further auxiliary points, lines, planes, and bodies. If not explicitely mentioned otherwise, these objects will continuously depend on the four points $P_L, \dots, P_B$ considered. Consequently, their existence, and certain further properties we are interested in, follow once they are established for the original quadrupel $\{P_{L_0}, \dots, P_{B_0}\}$. All together, after adding a fifth point to our configuration we will obtain a figure like presented in the picture below. Therefore, we will refer to parts of the figure as left arm, front leg, etc. For $X \in \{F, B\}$ and $Y \in \{R, L\}$ we select the rank-one lines d_{XY} in $\mathcal{HIP}$

trough P_X with direction close to $(d_1^X, d_2^Y, 2)$ where $d_1^F = -d_1^B = -\sqrt{3}$ and $d_{2^R} = -d_2^L = 1$. Similar, d_{YX} is the rank-one line in $\mathcal{HIP}$ trough P_Y with direction close to $(\tilde{d}_1^X, \tilde{d}_2^Y, -2)$, $\tilde{d}_1^F = -\tilde{d}_1^B = 1$ and $\tilde{d}_2^R = -\tilde{d}_2^L = -\sqrt{3}$. As all of these lines run in a 2-dimensional surface, we can in a continuous way define for c_1, c_2 the point $P_{c_1 c_2}$ as the intersection of the line $d_{c_1 c_2}$ with the line $d_{c_2 \bar{c}_1}$, where $\{c_1, \bar{c}_1\} \in \{\{F, B\}, \{R, L\}\}$ and $c_2 \in \{F, B, L, R\} \backslash \{c_1, \bar{c}_1\}$. These are precisely the (four plus eight) points already considered in the fourfold Tartar-configuration. Note that always $P_{\bar{c}_2, c_1} \in [P_{c_1}, P_{c_1 c_2}]$.

As our final configuration consists of five points, we have to break the symmetry kept until now. First, we choose a point $P_K \in \{p \in \mathbb{R}^3 : \langle e_2, p \rangle = 0\}$ close to $P_{K_0} = (0, 0, 2 - \sqrt{3})$ such that rank$(P_K - P_F)$=rank$(P_K - P_B) = 1$. The required continuous dependence is an easy consequence of the implicit function theorem. Further, we define two points in $\mathcal{HYP}$ by

$$\{R\} = d_{FR} \cap d_{BR}, \text{ close to } (0, 2/\sqrt{3}, 1/\sqrt{3}) \text{ and } \{L\} = d_{FL} \cap d_{BL},$$
$$\text{close to } (0, -2/\sqrt{3}, 1/\sqrt{3}).$$

Moreover, we set $\Delta = [R, L, P_K]$ - then P_F and P_B are strictly separated by the plane containing Δ. Hence, we can already define the two legs $\mathcal{L}_B = [P_B, R, L, P_K]$ and $\mathcal{L}_F = [P_F, R, L, P_K]$ with their feets P_B and P_F and their union $\mathcal{L} = \mathcal{L}_B \cup \mathcal{L}_F$. The description of the lower part of the figure is completed by setting for $X \in \{F, B\}$

- λ_X^∞ is the plane containing P_X, P_{XR} and P_{XL}, or equivalently containing P_X, R and L.
- λ_X^- is the halfspace below λ_X^∞, so more formally $\lambda_X^- = \{p + te_3 : p \in \lambda_X^\infty \text{ and } t < 0\}$.
- $\mathcal{L}_X^\infty = \bigcup_{s>0} P_X + s(\mathcal{L}_X - P_X)$ is the extension of the leg $\mathcal{L}_X$ to a cone.
- Note that $P_{\bar{X}} \in \lambda_X$ and also that the ray from P_X through P_K intersects $\lambda_{\bar{X}}^\infty$ in an inner point of $[P_{\bar{X}}, R, L]$.
- For later use we also observe that $P_{R\bar{X}}, P_{L,\bar{X}} \notin \mathcal{L}_X^\infty$, since $\{P_{Y\bar{X}}\} = d_{Y\bar{X}} \cap d_{\bar{X}\bar{Y}}$ and $d_{XY} \cap \mathcal{L}_X^\infty = \{\bar{Y}\}$.

Now, define the upper torso. For $Y \in \{L, R\}$ we set

- $\alpha_Y = [P_Y, P_{YF}, P_{YB}]$ and α_Y^∞ is the plane spanned by this triangle
- α_Y is again the halfspace below $\alpha_Y^\infty, \alpha_Y^+$ the one above.

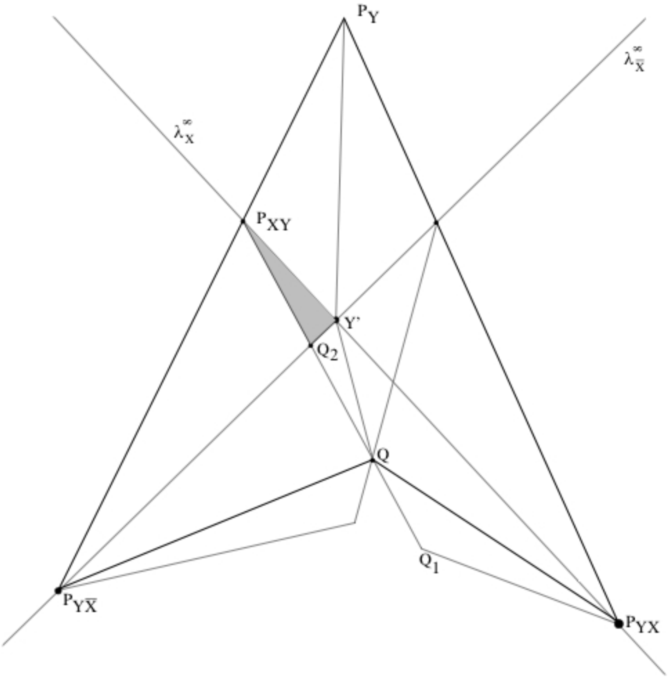
Then, given $Y \in \{L, R\}$ and $X \in \{F, B\}$ we denote by $\mathcal{S}_{XY} = \alpha_Y \cap \mathcal{L}_X^\infty$ the shoulder with upper corner P_{XY}.

Next, let L' be the intersection of α_L with $[L, R]$ and similar $\{R'\} = \alpha_R \cap [L, R]$. Both points are inner points of the segment $[L, R]$ continuously dependent on P_L, P_R, P_F, P_B. L'' is now the unique point on $[LR]$ and near $(0, -2 + \sqrt{3} - (1/\sqrt{3}), 1/\sqrt{3})$ which is rank-one connected to P_L, similar $R'' \in [LR]$ near $(0, 2 - \sqrt{3} + (1/\sqrt{3}), 1/\sqrt{3})$ is rank-one connected to P_R. Now we define for $X \in \{F, B\}$, $Y \in \{L, R\}$ the sets $\mathcal{A}_{YX} = [P_Y, Y', Y'', P_{YX}]$ and then we obtain the left and right arm setting $\mathcal{A}_Y = \mathcal{A}_{YF} \cup \mathcal{A}_{YB}$.

Before finishing our construction by adding the fifth point and the head coming with it, we establish some further geometrical properties of the parts already produced.

Lemma 5.1. *We have*

(1) $\mathrm{int}(\mathcal{L})=\mathrm{int}(\mathcal{L}_F \cup \mathcal{L}_B)=(\mathrm{int}(\mathcal{L}_F^\infty) \cap \lambda_B^-)\cup(\mathrm{int}(\mathcal{L}_B^\infty) \cap \lambda_F^-)$.
(2) For all ϵ positive and $Y \in \{L, R\}$ there is $\delta > 0$ such that $p \in \lambda_F^- \cap \lambda_B^-$, $\mathrm{dist}(p, \mathcal{A}_Y) < \delta$ and $\mathrm{dist}(p, \{P_{YF}, P_{YB}\}) > \epsilon$ implies $x \in \mathrm{int}(\mathcal{L})$.
(3) For all ϵ positive there is $\delta > 0$ such that $X \in \{F, B\}$, $Y \in \{L, R\}$ and $p \in \alpha_Y^+$ with $\mathrm{dist}(p, \{P_{XY}, P_{YX}\}) > \epsilon$ and $\mathrm{dist}(p, \mathcal{S}_{XY} \cap (\lambda_X^\infty \cup \lambda_{\bar{X}}^\infty \cup \lambda_{\bar{X}}^+)) < \delta$ implies $p \in \mathrm{int}(\mathcal{A}_\mathrm{Y} \cup \mathcal{L})$.



Corollary 5.2.

(1) *For each* $\epsilon > 0$ *there is* δ *positive such that any* $p \in \mathrm{int}(\mathcal{L})$ *with* $\mathrm{dist}(\mathrm{p}, [\mathrm{L}, \mathrm{R}] \cup \{\mathrm{P_F}, \mathrm{P_B}\}) > \epsilon$ *is the centre of a rank-one segment of lenght* δ *contained in int*$(\mathcal{L})$.
(2) *For any* $Y \in \{L, R\}$ *and* $\epsilon > 0$ *there is a* $\delta > 0$ *such that each* $p \in \mathrm{int}(\mathcal{A}_\mathrm{Y})$ *satisfying* $\mathrm{dist}(\mathrm{p}, \{\mathrm{P_Y}\} \cup [\mathrm{Y}', \mathrm{Y}'', \mathrm{P_{YF}}] \cup [\mathrm{Y}', \mathrm{Y}'', \mathrm{P_{YB}}]) > \epsilon$ *is the centre of a rank-one segment of lenght* δ *contained in* $\mathrm{int}(\mathcal{A}_\mathrm{Y})$.

We finish our construction by defining the head of the figure. Going back to our original configuration of points, we set $P_0 = (L_0' + R_0')/2 + \frac{1}{2}|L_0' - R_0'|e_3$. Clearly, P_0 is rank-one connected to L_0' and R_0'. Moreover, the rank-one connections from P_0 pass in L_0' and R_0' into $\mathrm{int}(\mathcal{L}_0)$. This shows that we can choose points P_{H_0} (near P_0) and $P_{1_0},\dots,P_{4_0}$ such that $\widehat{\det}(P_{H_0} - P_{j_0}) = 0$ if $j = 1,\dots,4$, that $[P_{1_0}\dots P_{4_0}] \subset \mathrm{int}(\mathcal{L}_0)$ and that $[L_0'R_0'] \in \mathrm{int}[P_{H_0}P_{1_0}\dots P_{4_0}]$. Now for a five point configuration $\mathcal{K} = \{P_L, P_R, P_F, P_B, P_H\}$ sufficiently close to $\{P_{L_0}, P_{R_0}, P_{F_0}, P_{B_0}, P_{H_0}\}$ we know that $\mathcal{L}$ is sufficiently close to $\mathcal{L}_0$ and hence, we can again find $P_1,\dots,P_4$ rank-one connected to P_H and such that $[P_1, P_2, P_3, P_4] \subset \mathrm{int}(\mathcal{L})$ and $[L', R'] \subset \mathrm{int}([\mathrm{P}_{\mathrm{H}}, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4])$. we denote by $\mathcal{H} = [[P_H, P_1, P_2, P_3, P_4]]$ the head of our figure.

After all this preparations, we will consider the open set

$$\mathcal{U} = \mathrm{int}(\mathcal{L} \cup \mathcal{A}_R \cup \mathcal{A}_L \cup \bigcup_{X=F,B} \bigcup_{Y=L,R} \mathcal{S}_{XY} \cup \mathcal{H})$$

together with our set $\mathcal{K}$ and show that symmetric gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. it can be checked that $\mathcal{U}$ is not starshaped with respect to any point, indeed a quite straightforward calculation gives that points above the bottoms of both legs are also above the tops of the two arms. We have that gradients are stable only near $\mathcal{K}$.

Theorem 5.3. *Consider a pair $\mathcal{U}, \mathcal{K}$ as obtained in the construction given above. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that $\Phi_{\mathcal{U}}(p) > \delta$ if $p \in \mathcal{U}$ and $\mathrm{dist}(p, \mathcal{K}) > \epsilon$.*

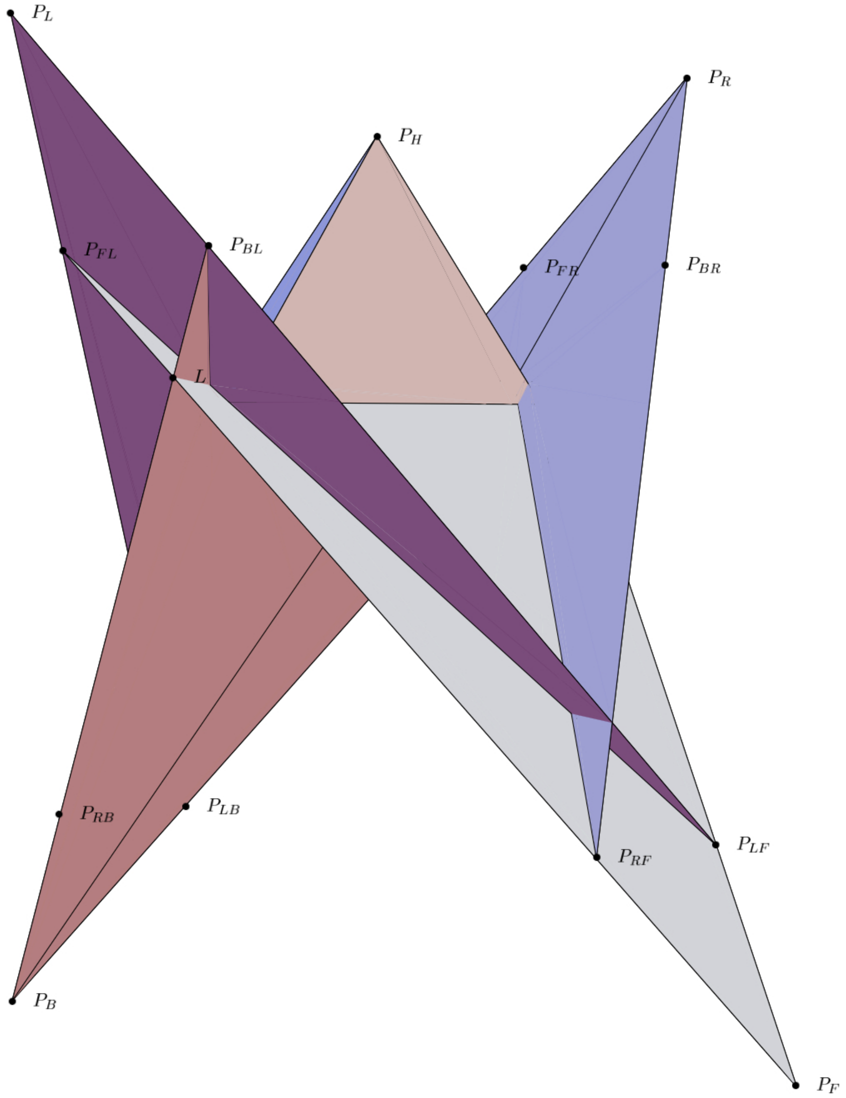
Remark 5.1. The definition of the function $\Phi_{\mathcal{U}}$ can be found in [Kir], Definition 3.19.

As a corollary we get,

Corollary 5.4. *There are five matrices $P_{F_0}, P_{B_0}, P_{L_0}, P_{R_0}, P_{H_0}$ in $\mathbb{M}^{2\times 2}_{sym}$ and an $\epsilon > 0$ such that for arbitrary*

$$P_1 \in B(P_{F_0}, \epsilon), P_2 \in B(P_{B_0}, \epsilon), P_3 \in B(P_{L_0}, \epsilon),$$
$$P_4 \in B(P_{R_0}, \epsilon), P_5 \in B(P_{H_0}, \epsilon)$$

in $\mathbb{M}^{2\times 2}_{sym}$ is $\{P_1, \dots, P_5\}$ a non-rigid set without rank-one connections.



Let ∇u be symmetric and $\det(\nabla u) \equiv -a \leq 0$. Consider

$$u_\epsilon^\pm(z) = u(z) \pm \sqrt{a+\epsilon}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} z. \tag{5.1}$$

It holds that

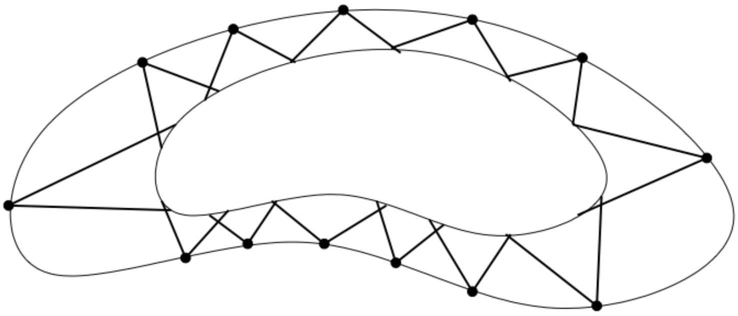
$$\det(\nabla u_\epsilon^\pm) \equiv \epsilon > 0,$$

and since $u_\epsilon^\pm$ is quasiregular and open we have that $\partial u_\epsilon^\pm(S) \subset u_\epsilon^\pm(\partial S)$ for every set $S \subset \Omega$.
Now, if $\epsilon \to 0$, then $u_0^\pm(\bar{S}) = \partial u_0^\pm(S) \subset u_0^\pm(\partial S)$ and since $\mathrm{Leb}(u_0^\pm(\bar{S})) = 0$, then we obtain $u_0^\pm(\Omega) = u_0^\pm(\partial\Omega)$ is a Lipschitz curve and therefore $(u_0^\pm)^{-1}(z)$ curves up to boundary.

Hence

$$\nabla u_0(p) \simeq \mathrm{Tan}_{u_0(p)} u_0(\Omega) \otimes \perp_p u_0^{-1}(u_0(p))$$

Generic large finite sets contain no R1C, but are not rigid.
For all U open bounded, $\forall\eta > 0$, $\exists\gamma > 0$ such that $\forall\mathcal{K}$ γ-dense in ∂U there is V open, $V \supset \{x; B(x,\eta) \subset U\}$ and gradients in V are stable only near $\mathcal{K}$.



Hence, $\mathcal{K}$ can be (made) finite, kind of reduction principle for non-rigid sets. $(\partial U \to \mathcal{K})$

6. **Solving 2nd order PDE by 1st order PDI.**

In this section we make a review or the paper [Sz]. In that paper the regularity of critical points for functionals of the type

$$I[u] = \int_{\Omega} F(Du)\,dx \tag{6.1}$$

is considered. More precisely, a smooth, strongly polyconvex $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and Lipschitzian but not C^1 weak solutions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ to the corresponding Euler-Lagrange system are constructed. F can be chosen in a such a way that these solutions are in fact weak local minimisers.

Let $\Omega \subset \mathbb{R}^2$ be the unit ball. As we said, we study the critical points of the (6.1), where $u : \Omega \to \mathbb{R}^2$ and $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is a smooth function with bounded second derivatives. The Euler-Lagrange equations can be written as

$$\operatorname{div} DF(Du) = 0. \tag{6.2}$$

In [MŠ], Müller & Šverák constructed a *strongly quasiconvex* F so that the corresponding system (6.2) admits weak solutions that are Lipschitz but not C^1 in any open subset of Ω. Their method is based in a modification of M. Gromov's convex integration combined with ideas originating from Tartar's programme of compensated compactness.

We say that a function $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is strongly quasiconvex is for some $\gamma > 0$ the inequality

$$\int_{\mathbb{T}^n} F(X + D\eta) - F(X)\,dx \geq \gamma \int_{\mathbb{T}^n} |D\eta|^2\,dx$$

holds for all $X \in \mathbb{R}^{m\times n}$ and all periodic Lipschitz mappings $\eta : \mathbb{T}^n \to \mathbb{R}^m$. Due to Evans, we know that global minimisers of functionals $\int_{\Omega} F(Du)\,dx$ with strongly quasiconvex integrand are smooth outside a closed subset of Ω of Lebesgue measure zero. This result was extended by Kristensen & Taheri to the case of strong local minimisers. In the paper [Sz] the main result is the following one.

Theorem 6.1. *Let Ω be the unit ball in $\mathbb{R}^2$. There exists a smooth, strongly polyconvex function $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ with bounded second derivatives, such that the corresponding 2×2 elliptic system*

$$\operatorname{div} DF(Du) = 0$$

admits weak solutions $u : \Omega \to \mathbb{R}^2$, which are Lipschitz but not C^1 in any open subset of Ω. Moreover, F can be chosen so that these weak solutions are weak local minimisers of the corresponding functional $I[u] = \int_\Omega F(Du)\,dx$.

A function is said to be polyconvex if it is a convex function of the minors. More precisely,

Definition 6.1. A function $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is said to be *strongly polyconvex* if there exists a convex function $G : \mathbb{R}^5 \to \mathbb{R}$ and $\epsilon > 0$ so that

$$F(X) = \epsilon |X|^2 + G(X, \det X).$$

Remark 6.1. Polyconvexity is a commonly used structural assumption in mathematical models of elasticity. It is strictly stronger than quasiconvexity. Also remark that if F is uniformly convex, weak solutions of (6.2) are smooth.

The strategy in order to prove theorem (6.1) will be the one followed by Müller & Šverák. Under the hypothesis that there exists a T_N configuration in a certain set of matrices arising from the partial differential equation, and assuming a certain non-degeneracy (condition (C)), Lipschitz weak solutions can be constructed to the PDE that are nowhere C^1.

The difficulty is finding a function F that satisfies the requirement of polyconvexity and allows for the construction to be carried out.

6.1. T_N *configurations.*

Whether or not weak solutions to the PDE (6.2) can be constructed via convex integration (nowhere C^1) depends on geometrical-combinatorial properties of the mapping $X \to DF(X)$. In this section we recall the relevant definitions and results regarding *rank-one convexity*. A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank-one convex if f is convex along each rank-one line.

Definition 6.2. For a compact set $K \subset \mathbb{R}^{m\times n}$ we define the *rank-one convex hull* of K as

$$\mathcal{K}^{rc} = \{X \in \mathbb{R}^{m\times n} : f(X) \le \sup_K f \quad \forall f : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ rank-one convex}\}$$

and for general sets

$$E^{rc} := \bigcup_{K\subset E \text{ compact}} K^{rc}.$$

The objects dual to rank-one convex functions are a subclass of probability measures supported on $\mathbb{R}^{m\times n}$ called *laminates*. A probability measure ν on the space of $m \times n$ matrices is a laminate if

$$\langle \nu, f\rangle \geq f(\bar{\nu}) \quad \text{for all rank-one convex} \quad f : \mathbb{R}^{m\times n} \to \mathbb{R},$$

where $\bar{\nu}$ denotes the barycentre of the measure ν.

In view of the applications to elliptic PDEs, it is of fundamental importance that the rank-one convex hull of a set K can be nontrivial (strictly larger than K) even if K contains no rank-one connections, that is $\operatorname{rank}(X - Y) > 1$, for any two distinct $X, Y \in K$. This can be illustrated on an example consisting of four diagonal matrices:

$$X_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$
$$X_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

The important property is the following cyclic structure:

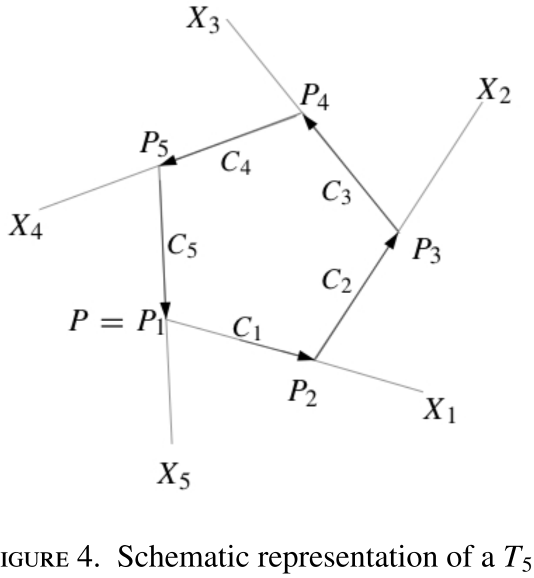


FIGURE 4. Schematic representation of a T_5.

Definition 6.3 (T_N configuration)**.** An ordered set of $N \geq 4$ matrices $\{X_i\}_{i=1}^N \subset \mathbb{R}^{m\times n}$ without rank-one connections is said to form a T_N configuration if there exists matrices $P, C_i \in \mathbb{R}^{m\times n}$ and real numbers $\kappa_i > 1$ such that

$$
\begin{aligned}
X_1 &= P + \kappa_1 C_1 \\
X_2 &= P + C_1 + \kappa_2 C_2 \\
&\vdots \\
X_N &= P + C_1 + \cdots + C_{N-1} + \kappa_N C_N,
\end{aligned}
\tag{6.3}
$$

and moreover $\operatorname{rank}(C_i) = 1$ and $\sum_{i=1}^N C_1 = 0$.

For a example a T_5 configuration can be represented as in Fig. 4.

We have the following lemma.

Lemma 6.2. *Let $\{X_1, \dots, X_N\}$ be a T_N configuration, and for $i = 1, \dots, N$ let $P_i = P + C_1 + \cdots + C_{i-1}$ (so that $P_1 = P$). Then*

$$\{P_1, \dots, P_N\} \subset \{X_1, \dots, X_N\}^{rc}.$$

In particular, for each $k = 1, \dots, N$ there exists numbers $\nu_i^{(k)} \in (0,1)$ such that the probability measures

$$\nu^{(k)} = \sum_{i=1}^N \nu_i^{(k)} \delta_{x_i}$$

are laminates with barycentre $\bar{\nu}^{(k)} = P_k$.

It is well known that T_N configurations form locally a manifold in the space of ordered $N - tuples$ of matrices . In the 2×2 case this manifold has the same dimension as the ambient space $(\mathbb{R}^{2\times 2})^N$, in other words T_N configurations are stable with respect to small perturbations. In higher dimensions this is no longer true, but we can find the right dimension for manifolds formed by T_N configurations. We state the necessary result for general $N \geq 4$ in $\mathbb{R}^{4\times 2}$.

Lemma 6.3 (Stability of T_N in $\mathbb{R}^{4\times 2}$). *Suppose the ordered set of matrices*

$$(X_1^0, \dots, X_N^0) \in (\mathbb{R}^{4\times 2})^N$$

is a T_N configuration. Then locally around $(X_1^0, \dots, X_N^0)$ there exists a smooth manifold $\mathcal{M}_N \subset (\mathbb{R}^{4\times 2})^N$ of dimension $6N$ such that all N-tuples

$$(X_1, \dots, X_N) \in \mathcal{M}_N$$

are T_N configurations.

6.2. ***Solutions by convex integration.***

We can rewrite the 2×2 system

$$\operatorname{div} DF(Du) = 0 \tag{6.4}$$

as a first-order differential inclusion. We note, that in two dimensions the divergence-free field $DF(Du)$ is a rotated curl-free field,

$$\operatorname{curl} DF(Du)J = 0, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the domain Ω is simply connected, there exists a potential $\tilde{u} : \Omega \to \mathbb{R}^2$ such that $DF(Du)J = D\tilde{u}$. If we denote by $\omega = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix}$ we see that 6.4 is equivalent to the inclusion

$$D\omega(x) \in K, \quad \text{with} \quad K = \left\{ \begin{pmatrix} X \\ DF(X)J \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\}. \tag{6.5}$$

Remark 6.2. K is a smooth 4-dimensional manifold in $\mathbb{R}^{2\times 2}$.

K depends on the function F, so in order to emphasise that we will also write K_F. If F satisfies the strong Legendre-Hadamard condition, then K_F is elliptic in the sense that the tangent space at any point contains no rank-one lines. The tangent space at a point is given by

$$T_{X_0}K = \left\{ \begin{pmatrix} X \\ D^2F(X_0)XJ \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\}.$$

Therefore $T_{X_0}K$ contains rank-one matrices if and only if there exist $a, b, n \in \mathbb{R}^2$ and $X_0 \in \mathbb{R}^{2\times 2}$ such that

$$D^2F(X_0)(a \otimes n)J = b \otimes n.$$

Using that $(a \otimes n)J = a \otimes n^{\perp}$, we get from the strong Legendre-Hadamard condition

$$0 < \left\langle D^2F(X_0)a \otimes n^{\perp}, a \otimes n^{\perp} \right\rangle = \langle b \otimes n, a \otimes n^{\perp} \rangle = 0,$$

which is a contradiction. In fact there are no rank-one connections in K.

Proposition 6.4. *Let $U \subset \mathbb{R}^{m\times n}$ be open and bounded, and let $A \in U^{rc}$. Then, for any $\delta > 0$ there exists a piecewise affine Lipschitz map $\omega : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\begin{aligned} D\omega(x) &\in U \quad \text{in}\, \Omega\, \text{a.e.}, \\ \omega(x) &= Ax \quad \text{on}\, \partial\Omega, \\ |\omega(x) - Ax| &< \delta \quad \text{in}\, \Omega. \end{aligned} \tag{6.6}$$

Moreover, if A is the barycentre of a laminate ν supported on a finite subset of U, with $\nu = \sum \nu_i \delta_{Z_i}$, then for each $\epsilon > 0$, ω can be chosen in addition so that

$$|\{x \in \Omega : \operatorname{dist}(D\omega(x), Z_i) < \epsilon\}| = \nu_i |\Omega|. \tag{6.7}$$

The key idea is to find enough T_N configurations in K so that the generate open sets and use Proposition 6.4 iteratively. It suffices to find just one T_N configuration and combine this with a transversality argument, which yields a submanifold of T_N configurations in K. We got from Lemma 6.3, that around an ordered N-tuple

$$(X_1^0, \ldots, X_N^0) \in (\mathbb{R}^{4\times 2})^N$$

which is a T_N configuration, there exists a smooth manifold of ordered N-tuples $\mathcal{M}_N$, consisting of T_N configurations and dim$\mathcal{M}_N$=6N. Let

$$\mathcal{K}_F = K_F \times \cdots \times K_F,$$

so that $\mathcal{K}_F$ is a $4N$-dimensional smooth manifold in $(\mathbb{R}^{4\times2})^N$. Now, we define the maps $\pi_k, \phi_k : \mathcal{M}_N \to \mathbb{R}^{4\times2}$ as

$$\pi_k(Z_1, \ldots, Z_N) = P_k, \quad \text{and} \quad \phi_k(Z_1, \ldots, Z_N) = Z_k$$

for $k = 1, \ldots, N$, where P_k is as in Lemma 6.2.

Let us now recall some basic facts about transversality. Suppose two smooth manifolds $\mathcal{M}$ and $\mathcal{K}$ embedded in $\mathbb{R}^d$ intersect at a point z. The intersection is said to be *transversal* if the tangent spaces at the point z satisfy

$$T_z\mathcal{M} + T_z\mathcal{K} = \mathbb{R}^d. \tag{6.8}$$

A direct consequence is that locally the intersection $\mathcal{M} \cap \mathcal{K}$ is a smooth manifold. Furthermore dim $\mathcal{M} \cap \mathcal{K}$=dim $\mathcal{M}$+dim $\mathcal{K} - d$.

In our case, if $\mathcal{M}_N$ and $\mathcal{K}_F$ intersect transversally, the intersection is manifold of dimension $2N$. As $N \geq 4$, we expect that the map π_k restricted to $\mathcal{M}_N \cap \mathcal{K}_F$ is a submersion. That is, the image under π_k of the intersection is an open set.

Definition 6.4 (Condition (C))**.** Suppose $F \in C^2(\mathbb{R}^{2\times2})$ is such that K_F contains a T_N configuration $\{Z_i\}$ and $\mathcal{M}_N$ is the manifold of T_N configurations given by Lemma 6.3. If $\mathcal{M}_N$ and $\mathcal{K}_F$ intersect transversally, and if for each $k = 1, \ldots, N$ the map

$$\pi_k(Z_1, \ldots, Z_N) = P_k$$

is a local submersion on $\mathcal{M}_N \cap \mathcal{K}_F$, then F is said to satisfy condition (C) at $\{Z_i\}$.

Proposition 6.5. *Suppose $F \in C^2(\mathbb{R}^{2\times2})$ is such that the associated manifod K given by* (6.5) *contains a T_N configuration $\{Z_1^0, \ldots, Z_N^0\}$ and suppose F satisfies condition* (C) *at $\{Z_i^0\}$. Let $P_0 \in \{Z_1^0, \ldots, Z_N^0\}^{rc}$. Then for any $\delta > 0$ there exists a Lipschitz map $\omega : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^4$ with the following properties:*

(a) *$D\omega(x) \in K \cap \left(\bigcup_{k=1}^N B_\delta(Z_k^0)\right)$ a.e. in Ω,*
(b) *In particular $u = (\omega_1, \omega_2)$ is a weak solution to 6.2,*
(c) *$\omega(x) = P_0x$ on $\partial\Omega$, and $|\omega(x) - P_0x| < \delta$ in Ω,*
(d) *$D\omega$ has essential oscillation of order 1 in any subdomain of Ω, so that ω is nowhere C^1.*

Corollary 6.6. *Assume, as in Proposition 6.5, that $F \in C^2(\mathbb{R}^{2\times2})$, that K_F contains a T_N configuration $\{Z_1^0, \ldots, Z_N^0\}$ with $Z_k^0 = \begin{pmatrix} X_k^0 \\ Y_k^0 \end{pmatrix}$, F satisfies condition* (C) *at $\{Z_i^0\}$, and in addition that $D^2F(X_k^0)$ is positive definite for each k.*

Then for sufficiently small $\delta > 0$, the map ω constructed in Proposition 6.5 is such that $u = (\omega_1, \omega_2)$ is a weak local minimiser of

$$\int_\Omega F(Du(x))\,dx.$$

Müller & Šverák constructed a strongly quasiconvex function F for which K_F contains a T_4 configuration. Then the calculated the tangent space to $\mathcal{M}_4$ at the point of intersection with $\mathcal{K}_{F_0}$ to prove that a suitable perturbation can move into the non-degenerate situation (C). We will show that (C) can be achieved in a general situation, for any T_N configuration. In view of this and the fact that small enough perturbations of strongly polyconvex functions remain strongly polyconvex, it is sufficient to exhibit one T_N configuration for one specific strongly polyconvex function to prove Theorem 6.1. We will do with $N = 5$.

6.3. ***Polyconvex examples.***

Instead of fixing a specific strongly polyconvex function and looking fot T_5's in the corresponding set K_F, we look for a specific T_5 which lies in K_F for some strongly polyconvex function F. The difference is computational: for the former, we have to solve 15 nonlinear equations in 25 variables, whereas the latter can be reduced to linear programming.

Lemma 6.7. *There exists a smooth, strongly polyconvex function $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and a T_5 configuration $\{Z_i\} \subset \mathbb{R}^{4\times 2}$ such that $\{Z_i\} \subset K_F$. Moreover $D^2F(X_i)$ is positive definite for each i, where $Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$.*

Example 6.1. Take

$$Z_1 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \\ 20 & 20 \\ 14 & 14 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 3 & 5 \\ -5 & -9 \\ 0 & -10 \\ 3 & -1 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 4 & 3 \\ -9 & -5 \\ -41 & 0 \\ -21 & 3 \end{pmatrix},$$

$$Z_4 = \begin{pmatrix} -3 & -3 \\ 8 & 9 \\ 54 & 72 \\ 30 & 41 \end{pmatrix}, \quad Z_5 = \begin{pmatrix} 0 & 0 \\ -1 & -2 \\ -18 & -36 \\ -11 & -22 \end{pmatrix}$$

The corresponding rank-one pentagon is

$$C_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 10 & 10 \\ 7 & 7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ -5 & 10 \\ -2 & -4 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 0 \\ -3 & 0 \\ -23 & 0 \\ -13 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -3 & -3 \\ 7 & 7 \\ 36 & 36 \\ 19 & 19 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 0 & 0 \\ -1 & -2 \\ -18 & -36 \\ -11 & -22 \end{pmatrix}$$

and $P = 0, k_1 = \cdots = k_5 = 2$.

6.4. *Stable embedding of* T_N.

The purpose of this section is to prove that if for a function F_0 there is a T_N configuration contained in K_{F_0}, then for certain small perturbations F of F_0 the same T_N configuration is contained in K_F in a stable way (i.e., condition (C) holds). The requirement that K_F contains the *same* T_N means that we are not dealing with any generic perturbation of F_0. Thus we need to carefully analyse the structure of the tangent $T\mathcal{M}_N$. On the other hand, once F is such that $\mathcal{K}_F$ and $\mathcal{M}_N$ intersect transversally, any small perturbation of F leads to F' with $K_{F'}$ still containing some (possible different) T_N configuration. We give the statement of the theorem.

Theorem 6.8. *Suppose $F_0 \in C^2(\mathbb{R}^{2\times 2})$ is such that K_{F_0} contains a T_N configuration. Then for any $\delta > 0$ there exists $F \in C^2(\mathbb{R}^{2\times 2})$ with $|D^2F - D^2F_0| < \delta$ and such that K_F contains the same T_N configuration and moreover F satisfies the non-degeneracy condition* (C).

7. Rank-one convexity and Ornstein's L^1-noninequalities.

In [Or], the starting point was the following question: Is there a universal constant K such that the following inequality

(7.1)

$$\int_{\mathbb{R}^2} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| dx\,dy \le K\left(\int_{\mathbb{R}^2} \left|\frac{\partial^2 f(x,y)}{\partial x^2}\right| dx\,dy + \int_{\mathbb{R}^2} \left|\frac{\partial^2 f(x,y)}{\partial y^2}\right| dx\,dy\right)$$

is true for all C^∞, f vanishing outside of the unit square. The answer to this question is yes if $f \in L^p$ for $p \in (1,\infty)$. For $f \in L^1$ the answer is negative as was shown by Ornstein. We recall the main theorem (see [Or], Theorem 1).

Theorem 7.1. *Let B, $D_1, \dots, D_L$ be a set of linearly independent linear homogeneous differential operators in n variables of degree m. For any $K > 0$ there is an f vanishing outside the unit cube and C^∞ in the whole n space such that $\int |Bf| > K$ and $\int |D_i f| < 1$, $1 \le i \le L$.*

This result was used to prove the nonsolvability of several basic PDE's as can be seen in [BB] and [Mc]. In [BB] the setting is the following one. Let us consider the equation

$$\operatorname{div} Y = f \quad \text{on}\, \mathbb{T}^d \tag{7.2}$$

i.e., they worked with 2π-periodic functions in all the variables. It is assumed that $d \ge 2$ and that

$$\int_Q f = 0 \tag{7.3}$$

where $Q = (0, 2\pi)^d$. The notations L^p, $W^{1,p}$, etc. refer to $L^p(\mathbb{T}^d)$, $W^{1,p}(\mathbb{T}^d)$, etc. They denote $L^p_{\#}$ the space of functions in L^p satisfying (7.3).

Clearly, (7.2) is an undetermined problem which admits many solutions. A standard way of looking for solutions of (7.2) is to look for a vector field Y satisfying the additional condition

$$\operatorname{curl} Y = 0,$$

i.e., one looks for a *special* Y of the form

$$Y = \nabla u.$$

Equation (7.2) becomes

$$\Delta u = f \tag{7.4}$$

and the standard L^p-regularity theory yields a solution $u \in W^{2,p}$ when $f \in L^p_\#$, $1 < p < \infty$. Consequently (7.2) has a solution $Y \in W^{1,p}$ for every $f \in L^p_\#$, $1 < p < \infty$. More precisely, the operator div: $W^{1,p} \to L^p_\#$ admits a right inverse which is a bounded linear operator $K : L^p_\# \to W^{1,p}$.

Therefore, three limiting cases are considered.

Case 1: **p=1**. It is well known that when $f \in L^1$ equation (7.4) does not necessarily admit a solution $u \in W^{2,1}$. However, one might still hope to have some solution Y of (7.2) in $W^{1,1}$, or at least in BV. This is not true. For some $f's$ in L^1, equation (7.2) has no solution in BV and not even in $L^{d/(d-1)}$.

Case 2: **p=∞**. It is well known that when $f \in L^\infty$, equation (7.4) does not necessarily admit a solution $u \in W^{2,\infty}$. Still hope to find a solution Y of (7.2) in $W^{1,1}$. This is not true. McMullen [Mc] has shown that for some $f's$ in L^∞ (even f continuous) equation (7.2) has no solution in $W^{1,\infty}$. This is proved using a duality argument an a "non-estimate" of Ornstein [Or].

Case 3: **p=d**. In [BB] they proved that given any $f \in L^d_\#$ there is some $Y \in L^\infty$ solving (7.2), as states the following proposition

Proposition 7.2. *Given any $f \in L^d_\#$ there exists some $Y \in L^\infty$ solving* (7.2) *(in the sense of distributions) with*

$$\|Y\|_{L^\infty} \le C(d)\|f\|_{L^d}. \tag{7.5}$$

Here, we reproduce the proof for the *Case 2* for $d = 2$ as appears in [BB], which is essentially similar to the one of McMullen [Mc]. We argue by contradiction.

Then, for every $f \in L^\infty$ there is a $Y \in W^{1,\infty}$ satisfying

$$\operatorname{div} Y = f - \fint f$$

and

$$\|Y\|_{W^{1,\infty}} \le C\|f\|_{L^\infty}.$$

Let ψ be a smooth function on $\mathbb{T}^2$ and set $g = \psi_{x_1x_2}$. Write

$$\int g_{x_1}Y_1 + g_{x_2}Y_2 = -\int gf = -\int \psi_{x_1x_1}Y_{1x_2} + \psi_{x_2x_2}Y_{2x_1}.$$

Consequently

$$\left|\int gf\right| \leq C(\|\psi_{x_1x_1}\|_{L^1} + \|\psi_{x_2x_2}\|_{L^1})\|f\|_{L^\infty}$$

and thus

$$\|g\|_{L^1} = \|\psi_{x_1x_2}\|_{L^1} \leq C(\|\psi_{x_1x_1}\|_{L^1} + \|\psi_{x_2x_2}\|_{L^1}).$$

This contradicts the non-inequality of Ornstein [Or] and completes the proof.

Bernd Kirchheim and Jan Kristensen show how Ornstein's result naturally fits in the general framework of convex integration and that it can be extended to certain nonlinear settings.

If we define by

$$\begin{aligned} S^m(\mathbb{R}^N;\mathbb{R}^M) &= \{\text{symmetric m-linear maps}\, u : \mathbb{R}^N \to \mathbb{R}^M\} \\ &= \text{m-th derivatives}\, u : \mathbb{R}^N \to \mathbb{R}^M, \end{aligned}$$

and given $\phi_0, \phi_1, \dots, \phi_k : S^m(\mathbb{R}^N;\mathbb{R}^M) \to \mathbb{R}$ linear, then Ornstein is equivalent to the following statement.
Let $\phi_0 \notin \text{Span}\{\phi_1, \dots, \phi_k\}$. Then $\forall k$, $\exists \varphi \in C_0^\infty$ such that

$$\|\phi_0(D^m\varphi)\|_{L^1} > k \sum \|\phi_j(D^m\varphi)\|_{L^1}.$$

Let us denote by

$$F = F_p^k(\phi_0, \phi_1, \dots, \phi_k)(X) \to -|\phi_0(X)|^p + k \sum |\phi_j(X)|^p.$$

Then the following are equivalent,

$$\begin{aligned} &\exists C\, \text{such that}\, \forall \varphi,\, \|\phi_0(D^m\varphi)\|_{L^p} \leq C \sum \|\phi_j(D^m\varphi)\|_{L^p} \\ &\exists k\, \text{such that}\, \forall \varphi \in C_0^\infty \int F(D^m\varphi) \geq 0. \end{aligned}$$

Basically, it means that $\exists k$ such that F_p^k is quasiconvex in 0. Therefore, given $\phi_0 \notin \text{Span}\{\phi_1, \dots, \phi_k\} \Rightarrow \forall k, F_p^k$ not convex in 0.

Theorem 7.3. *If f is rank-one convex and* 1*-homogeneous function then f is convex in* 0 *and in all rank-one matrices.*

References

[BB] Bourgain, J., Brezis, H., *on the equation div Y=f and application to control of phases*, J. Amer. Math. Soc **16**/2 (2003), 393–426.

[BJ] Ball, J. M., James, R. D., *Fine phase mixtures as minimizers of energy*, Arch. Rational Mech. Anal. **100** (1987), 13–52.

[CK] Chlebík, M., Kirchheim, B., *Rigidity for the four gradient problem*, J. Reine. Angew. Math. **551** (2002), 1–9.

[Kir] Kirchheim, B., *Rigidity and Geometry of microstructures*, Habilitation thesis, University of Leipzig, 2003.

[KP] Kirchheim, B., Preiss, D., *Construction of Lipschitz mappings having finitely many gradients without rank one connections.*

[KS] Kirchheim, B., Székelyhidi, Jr., L.,*On the gradient set of Lipschitz maps*, J. Reine Angew. Math. **625** (2008), 215–229.

[Mc] McMullen, C.T., *Lipschitz maps and nets in Euclidean space*, Geom. Funct. Anal. **8** (1998), 304–314.

[MŠ] Müller, S., Šverák, V., *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. Math. (2) **157** (2003), 715–742.

[Or] Ornstein, D., *A non-inequality for differential operators in the L_1 norm*, Arch. Rational Mech. Anal. **11** (1962), 40–49.

[Ped] Pedregal, P., *Laminates and microstructure*, Europ. J. Appl. Math. **4** (1993), 121–149.

[Sz] Székelyhidi, Jr., L., *The Regularity of Critical Points of Polyconvex Functionals*, Arch. Rational Mech. Anal. **172** (2004), 133–152.

[Tar] Tartar, L., *A note on separately convex functions (II)*, Note 18, Carnegie-Mellon University (1987).